# A Brief Walk in the Universe: A Pedagogical Introduction to Cosmology

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# Prefaces

- We will only scratch the surface, because cosmology in 1 hour is difficult…
- The goals of this lecture:
  - Have a (very) rough idea about cosmology
  - Have a view on the current progress in this field

# Attack of Astronomers

Time: 1 Gyr $= 10^9$ yr $\approx 3\times10^{16}$ s $\approx 10^{60}\ t_{\mathrm{Planck}}$

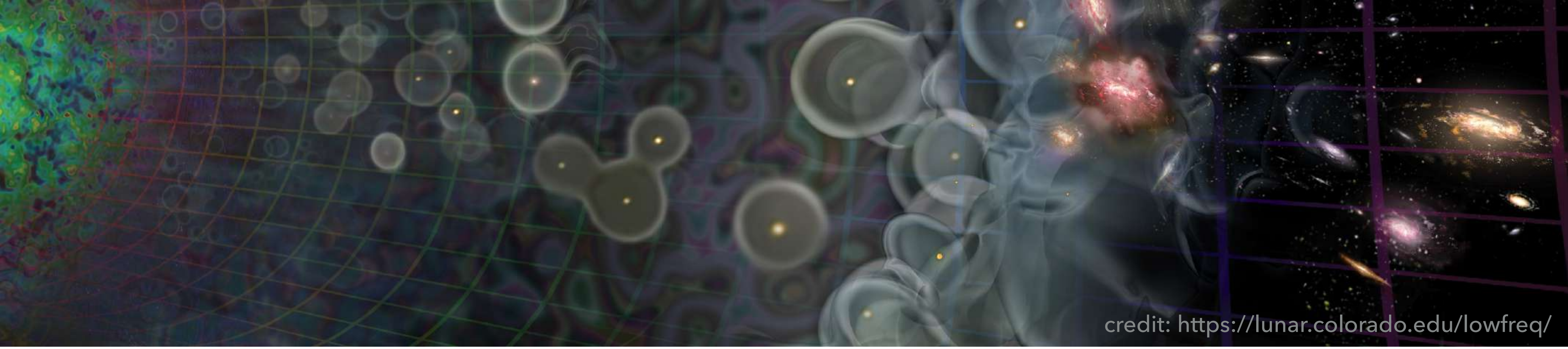
Age of the universe $\approx$ 14 Gyr

Distance: 1 Mpc $= 10^6$ pc $\approx 3\times10^{22}$ m $\approx 10^{57}\ d_{\mathrm{Planck}}$

Size of the universe $\approx$ 4000 Mpc

Mass: $1\ M_\odot \approx 2\times10^{30}$ kg $\approx 10^{38}\ m_{\mathrm{Planck}}$

Total mass of a galaxy $\approx 10^{12}\ M_\odot$




**What is cosmology?**

Cosmology is a study of the universe.

Cosmology is related to everything.

**Why do we study cosmology?**

Curiosity.

Cosmology always gives surprises (e.g., the cosmic expansion, dark matter and dark energy).

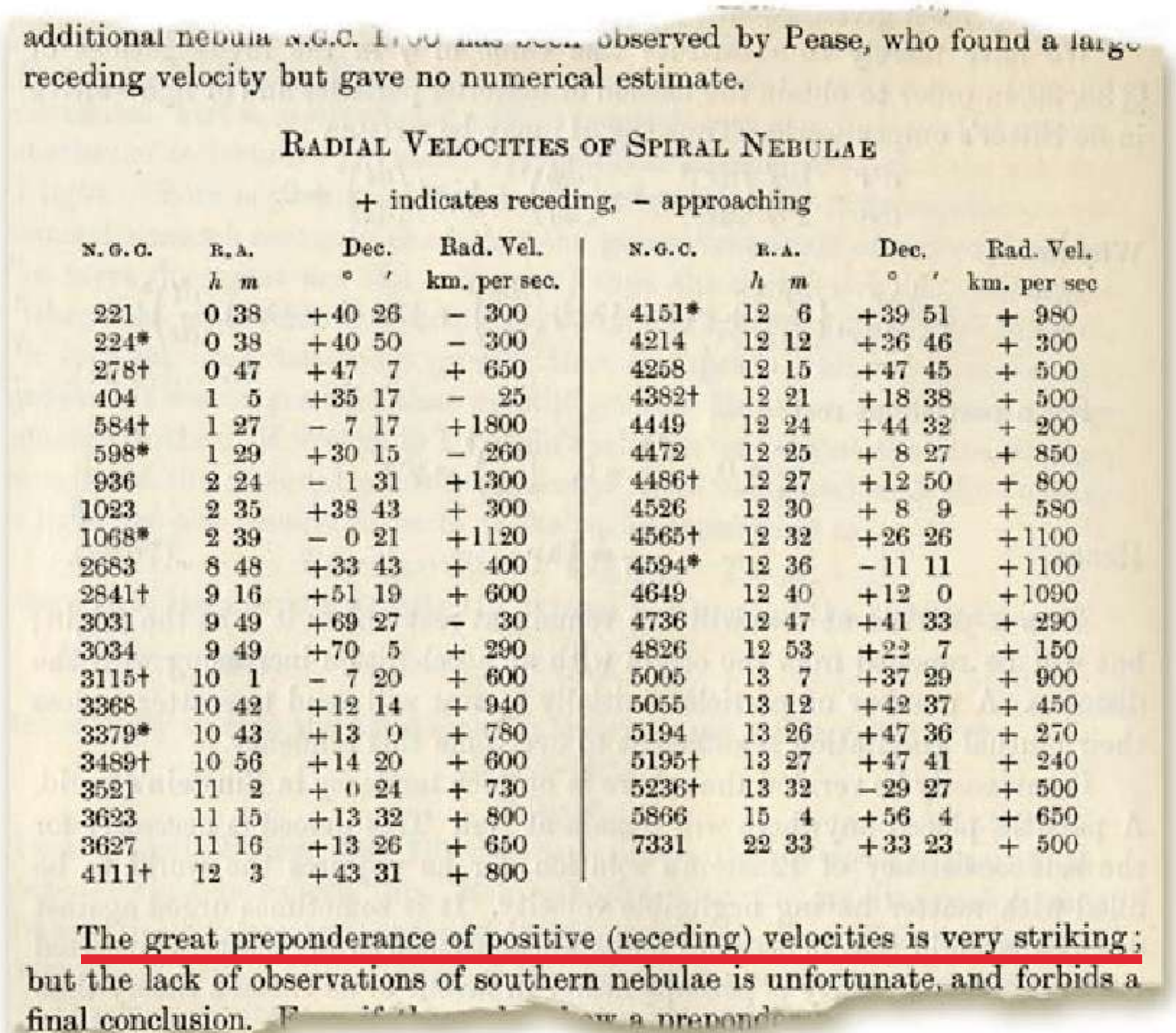
# Outlines

- ◉ The standard cosmological model
- ◉ The homogeneous universe
- ◉ The inhomogeneous universe
- ◉ Measurements of the universe

# The Standard Cosmological Model

# The Expansion of the Universe

- In 1923, Arthur Eddington compiled a list of wavelength shifts of 46 galaxies.
- 36 redshifting and 5 blueshifting.



additional nebula N.G.C. [illegible] observed by Pease, who found a large receding velocity but gave no numerical estimate.

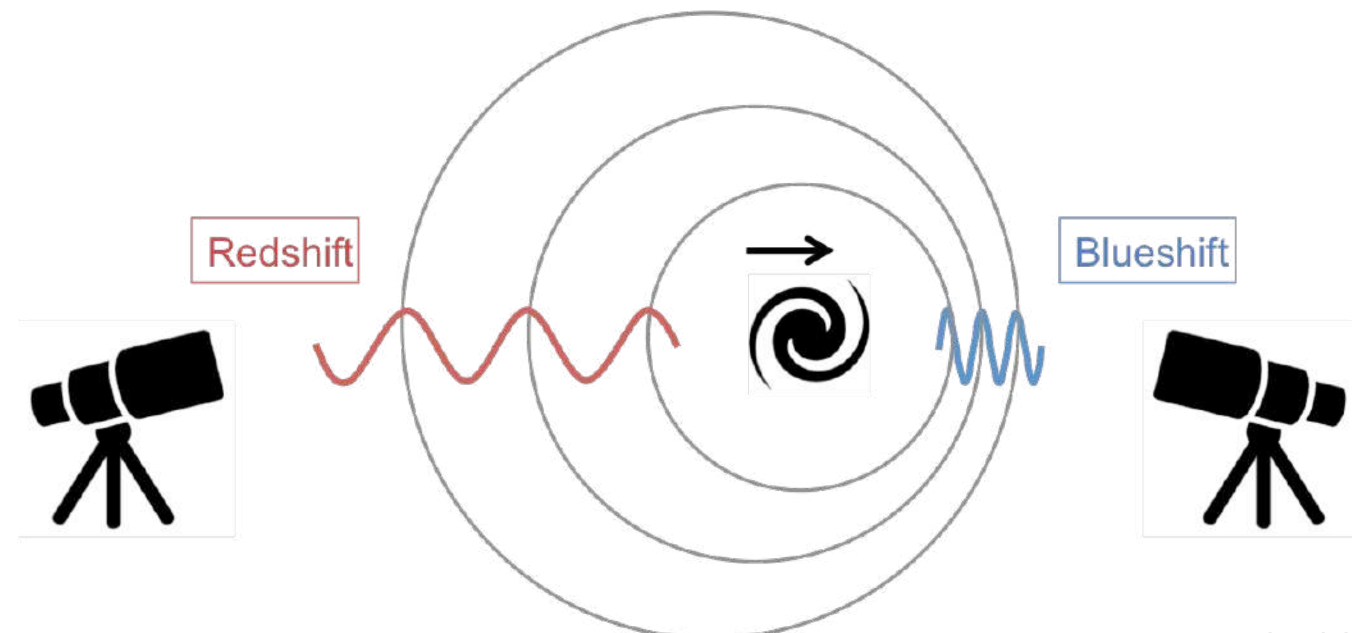
RADIAL VELOCITIES OF SPIRAL NEBULAE

\+ indicates receding, − approaching

| N.G.C. | R.A. | Dec. | Rad. Vel. | N.G.C. | R.A. | Dec. | Rad. Vel. |
|---|---|---|---|---|---|---|---|
| | h m | ° ′ | km. per sec. | | h m | ° ′ | km. per sec |
| 221 | 0 38 | +40 26 | − 300 | 4151* | 12 6 | +39 51 | + 980 |
| 224* | 0 38 | +40 50 | − 300 | 4214 | 12 12 | +36 46 | + 300 |
| 278† | 0 47 | +47 7 | + 650 | 4258 | 12 15 | +47 45 | + 500 |
| 404 | 1 5 | +35 17 | − 25 | 4382† | 12 21 | +18 38 | + 500 |
| 584† | 1 27 | − 7 17 | +1800 | 4449 | 12 24 | +44 32 | + 200 |
| 598* | 1 29 | +30 15 | − 260 | 4472 | 12 25 | + 8 27 | + 850 |
| 936 | 2 24 | − 1 31 | +1300 | 4486† | 12 27 | +12 50 | + 800 |
| 1023 | 2 35 | +38 43 | + 300 | 4526 | 12 30 | + 8 9 | + 580 |
| 1068* | 2 39 | − 0 21 | +1120 | 4565† | 12 32 | +26 26 | +1100 |
| 2683 | 8 48 | +33 43 | + 400 | 4594* | 12 36 | −11 11 | +1100 |
| 2841† | 9 16 | +51 19 | + 600 | 4649 | 12 40 | +12 0 | +1090 |
| 3031 | 9 49 | +69 27 | − 30 | 4736 | 12 47 | +41 33 | + 290 |
| 3034 | 9 49 | +70 5 | + 290 | 4826 | 12 53 | +22 7 | + 150 |
| 3115† | 10 1 | − 7 20 | + 600 | 5005 | 13 7 | +37 29 | + 900 |
| 3368 | 10 42 | +12 14 | + 940 | 5055 | 13 12 | +42 37 | + 450 |
| 3379* | 10 43 | +13 0 | + 780 | 5194 | 13 26 | +47 36 | + 270 |
| 3489† | 10 56 | +14 20 | + 600 | 5195† | 13 27 | +47 41 | + 240 |
| 3521 | 11 2 | + 0 24 | + 730 | 5236† | 13 32 | −29 27 | + 500 |
| 3623 | 11 15 | +13 32 | + 800 | 5866 | 15 4 | +56 4 | + 650 |
| 3627 | 11 16 | +13 26 | + 650 | 7331 | 22 33 | +33 23 | + 500 |
| 4111† | 12 3 | +43 31 | + 800 | | | | |

The great preponderance of positive (receding) velocities is very striking; but the lack of observations of southern nebulae is unfortunate, and forbids a final conclusion.

Eddington 1923




images: SDSS

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}}$$

$\lambda_{\rm obs}$ : Observed frame

$\lambda_{\rm emit}$: Rest frame

$z < 0$ **Blueshift**

$z > 0$ **Redshift**

The chance is less than $1/10^6$

# The Expansion of the Universe

- In 1929, Edwin Hubble measured the distance to the galaxies.
- The receding velocity has proportionality to the distance.
- The universe must be expanding
  - The farther distance, the higher receding velocity
  - Isotropy (no preferential direction)
- The beginning of modern cosmology

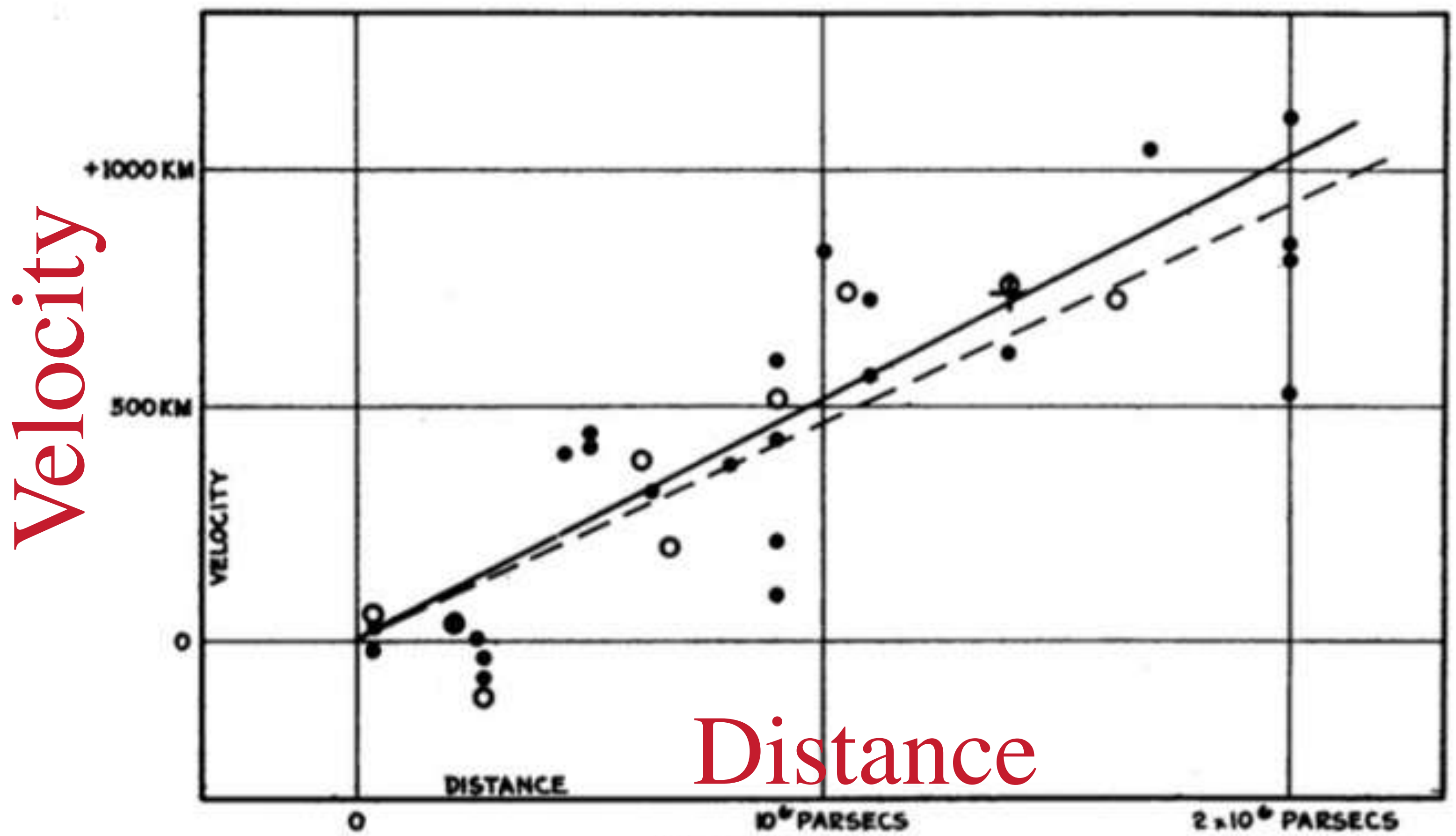
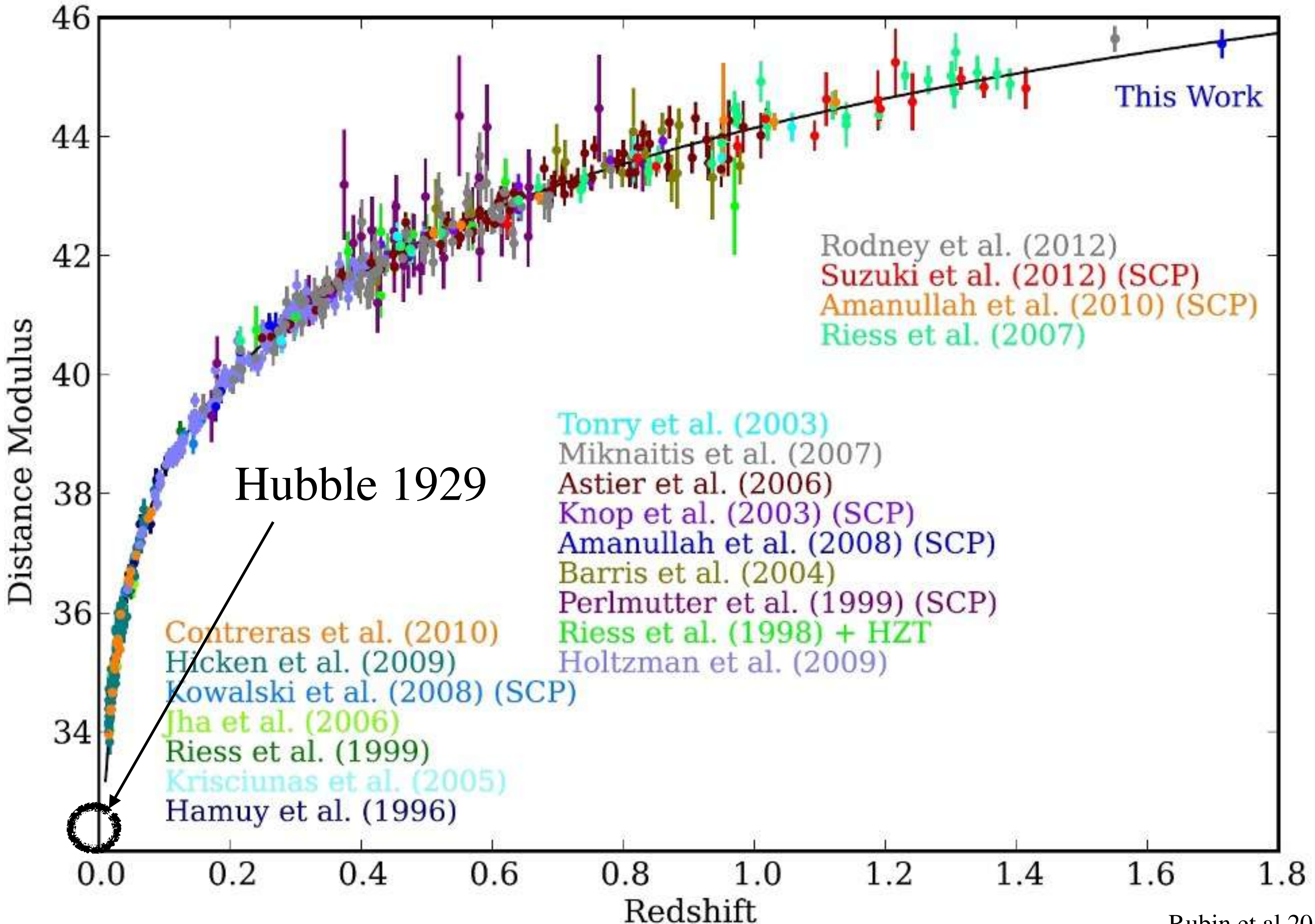



FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.
Hubble 1929




Rubin et al 2013

# Hubble–Lemaître law (or Hubble Law)

$$cz \equiv v = H_0 r$$

Redshift Distance

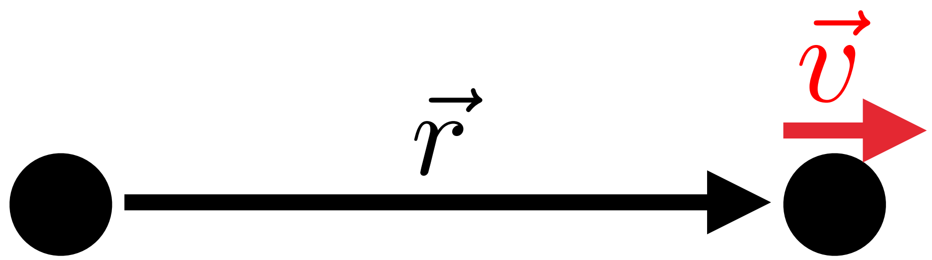
$$H_0 \equiv \text{Hubble constant} \approx h \times 100\,\frac{\text{km/sec}}{\text{Mpc}},\ \ h \approx 0.7$$

$$t_{\rm H} \equiv \frac{1}{H_0} \equiv \text{Hubble time} = \frac{r}{v} \approx 14\ \text{Gyr}$$

$$d_{\rm H} \equiv c\frac{1}{H_0} \equiv \text{Hubble distance} \approx 4400\ \text{Mpc}$$

**Distance measurements quantify the cosmic expansion.**

# Hubble–Lemaître law (or Hubble Law)

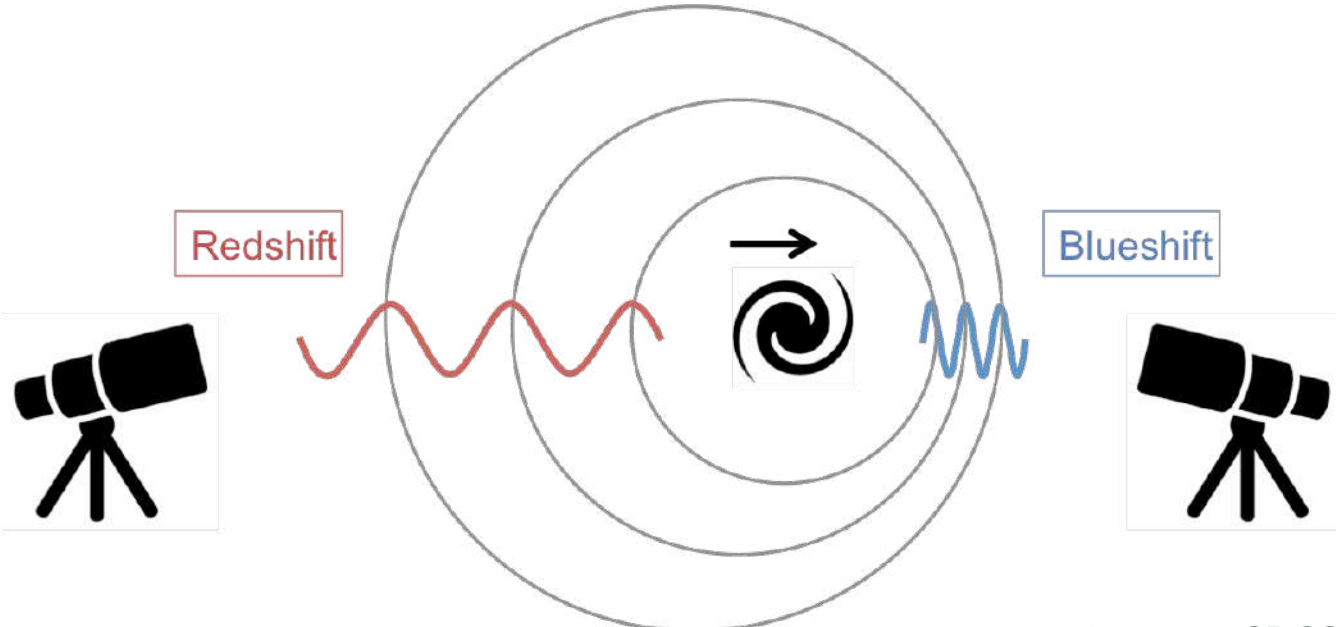



$$
\begin{aligned}
r &= a(t)\chi \\
a(t) &= \text{Scale factor with } a(t_0) = 1 \\
\chi &= \text{Comoving coordinate} = r(t_0) \\
v &\equiv \frac{\mathrm{d}r}{\mathrm{d}t} = \dot{a}\chi = \frac{\dot{a}}{a} r \\
H(t) &\equiv \text{Hubble parameter} = \frac{\dot{a}(t)}{a(t)} \\
H(t_0) &= \text{Hubble constant} \approx 70 \frac{\mathrm{km/sec}}{\mathrm{Mpc}}
\end{aligned}
$$

Hubble Law is a natural result of the homogeneous and isotropic expansion.

- **Homogeneous**: a(t) does not depend on $\vec{r}$.
- **Isotropic**: a(t) is a scalar.

# Redshift




images: SDSS

$t_e$

$$\lambda_e = a_e L$$

$t_o$

$$\lambda_o = a_o L$$

$$z \equiv \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}}$$

$\lambda_{obs}$ : Observed frame

$\lambda_{emit}$: Rest frame

$z < 0$ **Blueshift**

$z > 0$ **Redshift**

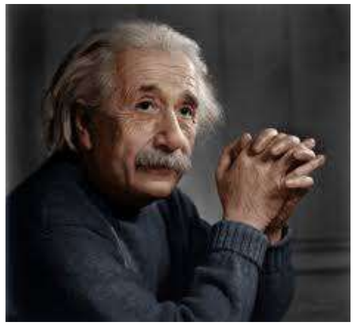
$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{a_o L - a_e L}{a_e L} = \frac{a_o}{a_e} - 1 \equiv \frac{1}{a} - 1$$

$$\boxed{\frac{1}{a} = 1 + z}$$

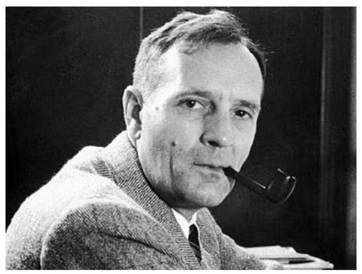
z = 0: now (or local universe)

z > 0: past (or distant universe)

# The Cosmological Models



In 1915, A. Einstein developed the "Einstein equations".
In his early model, he believed a "static" universe.



In 1929, E. Hubble discovered the expansion of the universe.

## The big bang model

The universe is homogeneous and isotropic (i.e., the cosmological principle) with the global properties (e.g., temperature, density) changing with time.

G. Gamor

G. Lemaître

## The steady state model

The universe is homogeneous and isotropic (i.e., the cosmological principle) with the constant global properties (e.g., temperature, density).

Bondi, Gold & Hoyle 1948

# The Steady State Cosmological Model

- Hubble Law does not require a big bang.
- The steady state model implies the creation of matter.

**Constant cosmic expansion**

$$\frac{\mathrm{d}r}{\mathrm{d}t} = H_0 r \Rightarrow r \propto e^{H_0 t} \text{ , assuming } H(t) = H_0.$$

**Constant mean matter density**

$$V \propto r^3 = e^{3H_0 t} \Rightarrow \dot{M} = \rho \dot{V} = \rho 3 H_0 V$$

$$\frac{\dot{M}}{V} = \rho 3 H_0 = \rho_0 3 H_0 \text{ (assuming } \rho = \rho_0) \approx 10^{-27} \frac{\mathrm{kg}}{\mathrm{m}^3} / \mathrm{Gyr}$$

$\boxed{\approx \textbf{1 atom in 1 m}^3\textbf{ per Gyr}}$

# The Cosmological Models

## The big bang model

The universe is homogeneous and isotropic (i.e., the cosmological principle) with the global properties (e.g., temperature, density) changing with time.

## The steady state model

The universe is homogeneous and isotropic (i.e., the cosmological principle) with the constant global properties (e.g., temperature, density).

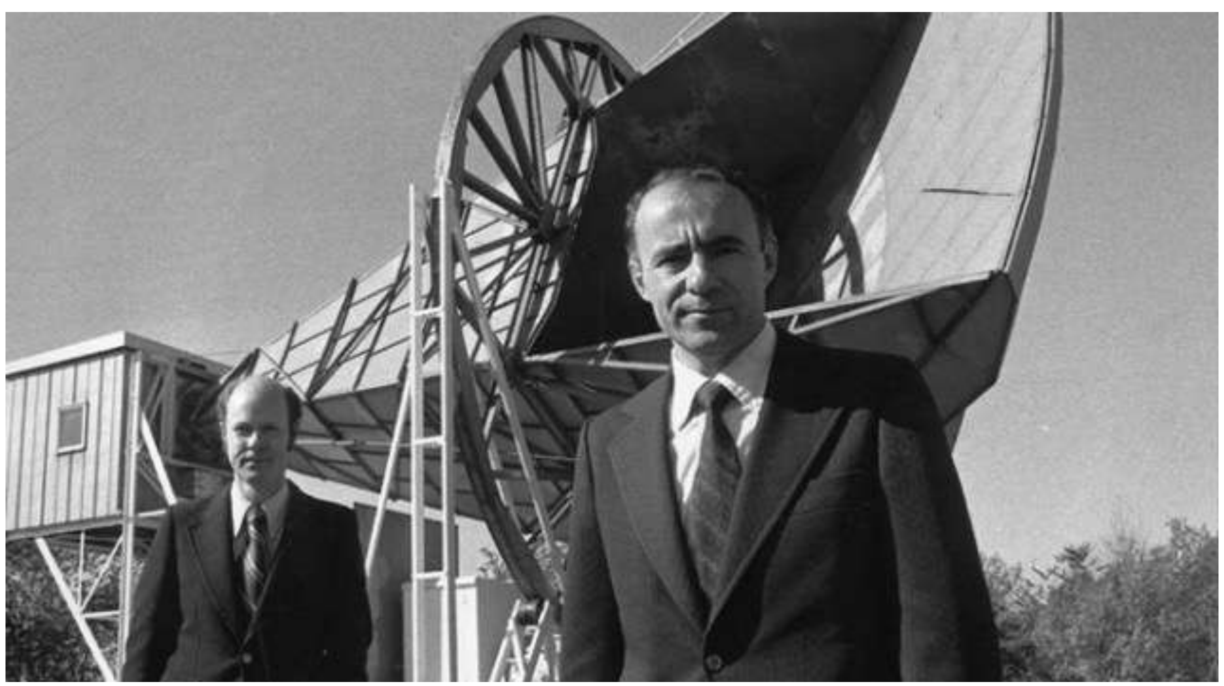
# The Cosmological Models

**The big bang model**

The universe is homogeneous and isotropic (i.e., the cosmological principle) with the global properties (e.g., temperature, density) changing with time.

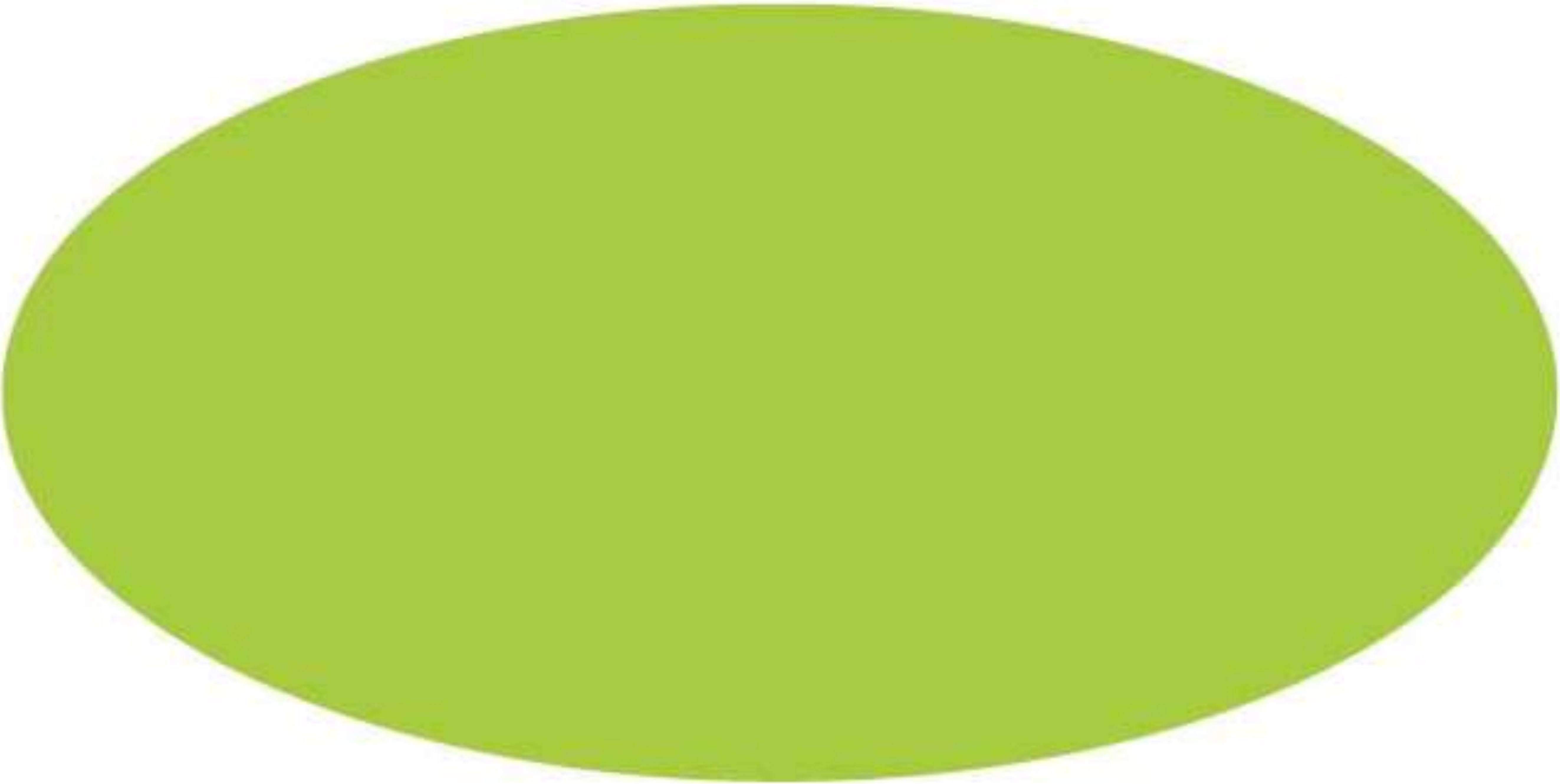
~~**The steady state model**~~

~~The universe is homogeneous and isotropic (i.e., the cosmological principle) with the constant global properties (e.g., temperature, density).~~



In 1965, Arno Penzias and Robert Wilson (accidentally) discovered the "**Cosmic Microwave Background (CMB)**", which is evidence of the hot big bang.

# The Cosmic Microwave Background (CMB)

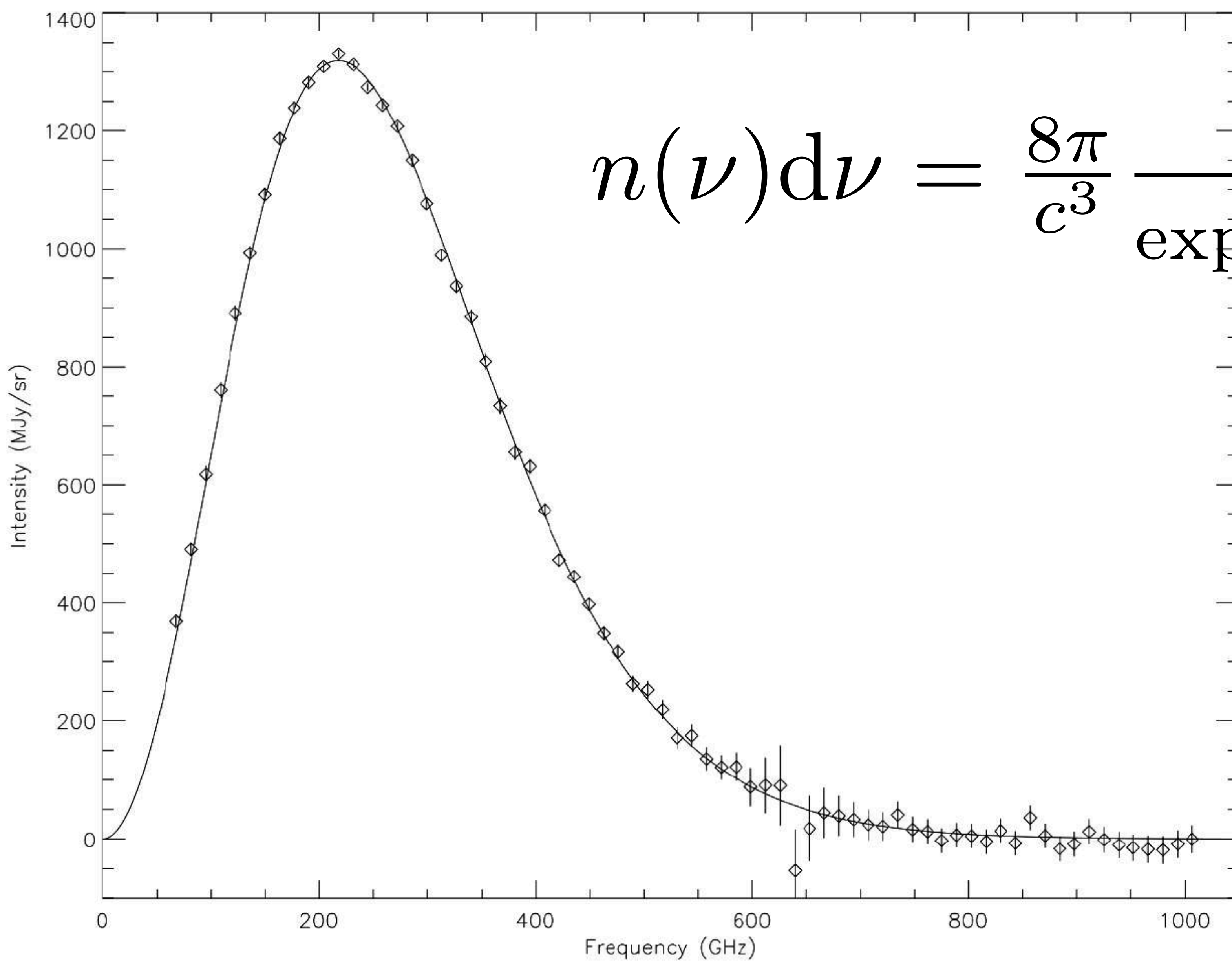


NASA

Isotropic 2.7K

# The Cosmic Microwave Background (CMB)

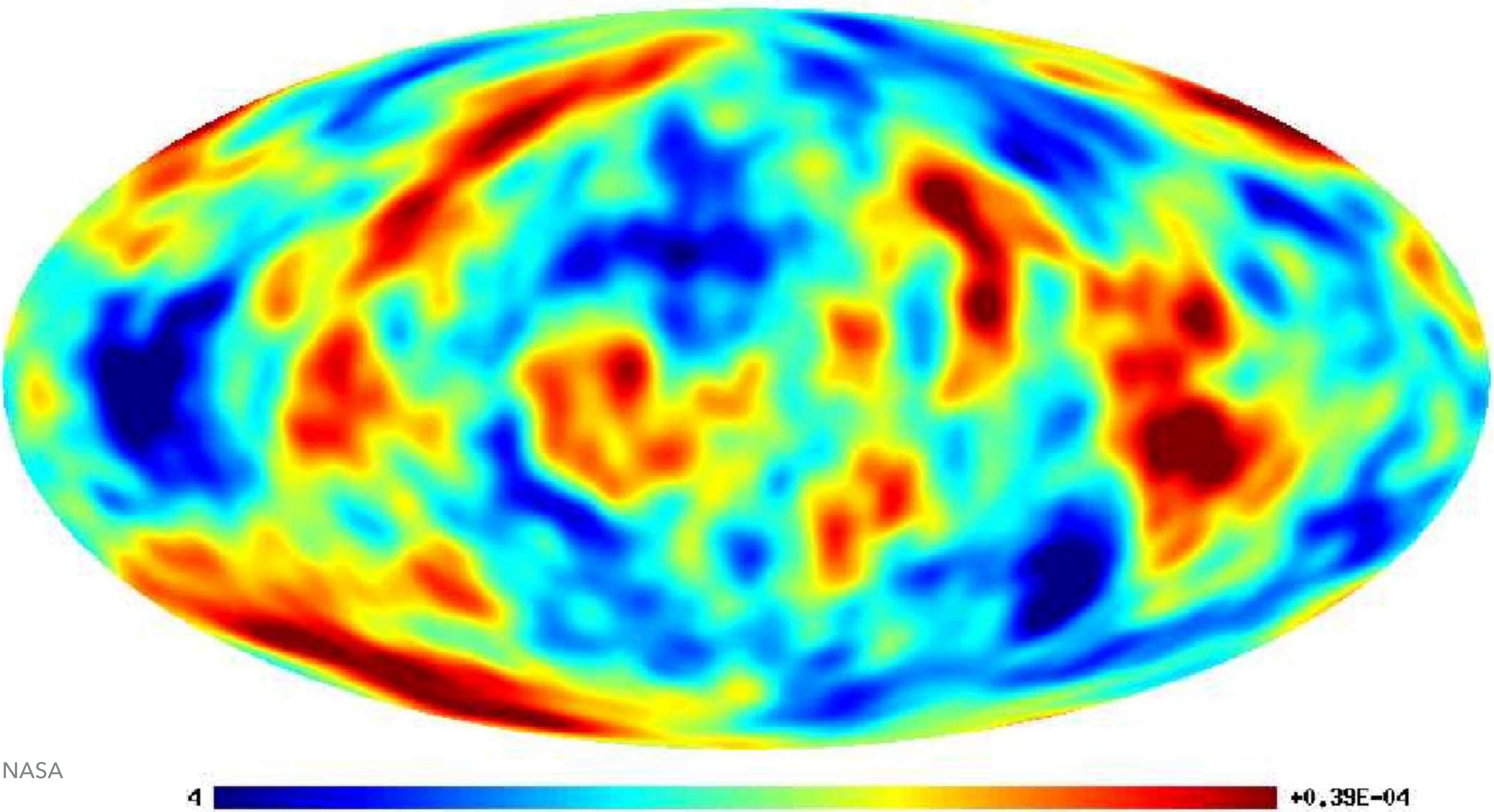
- CMB is excellently fitted by a blackbody radiation with T = 2.72548 ± 0.00057 K.
- The blackbody CMB ⇒ a thermal equilibrium ⇒ high collision rates of photons ⇒ the universe is opaque.
- The global properties change with time (opaque → transparent).
- The CMB is **the relic of the hot big bang**.




$$n(\nu)\mathrm{d}\nu = \frac{8\pi}{c^3}\frac{\nu^2\mathrm{d}\nu}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1}$$

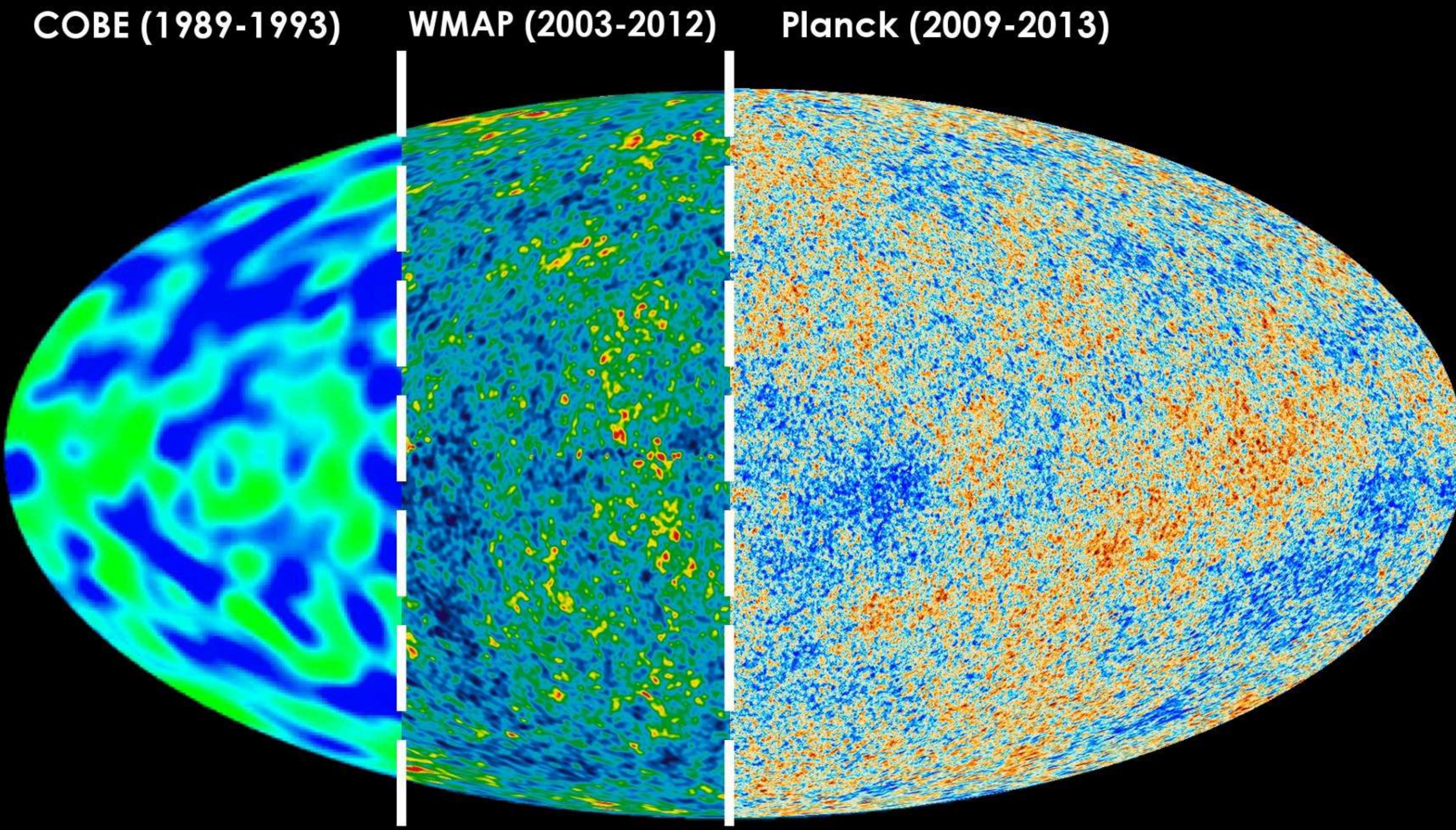
Fixten+09

# The Anisotropy of CMB at Small Scales




$$|\frac{\delta T}{T}| \approx 10^{-5} \text{ for } 5 \ \text{deg} \lesssim \theta \lesssim \ 180 \ \text{deg}$$

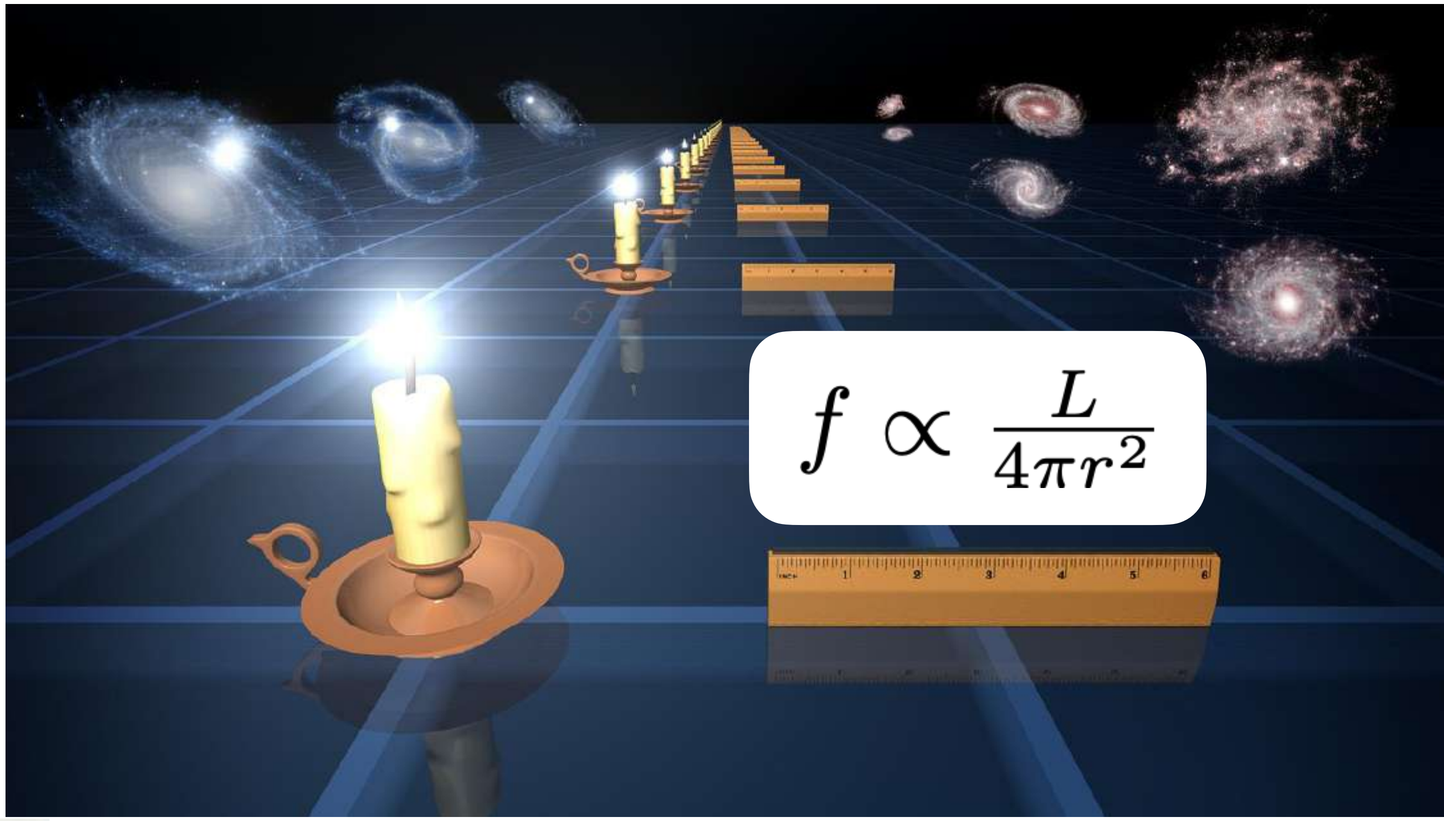
# The Triumph of Cosmology—CMB

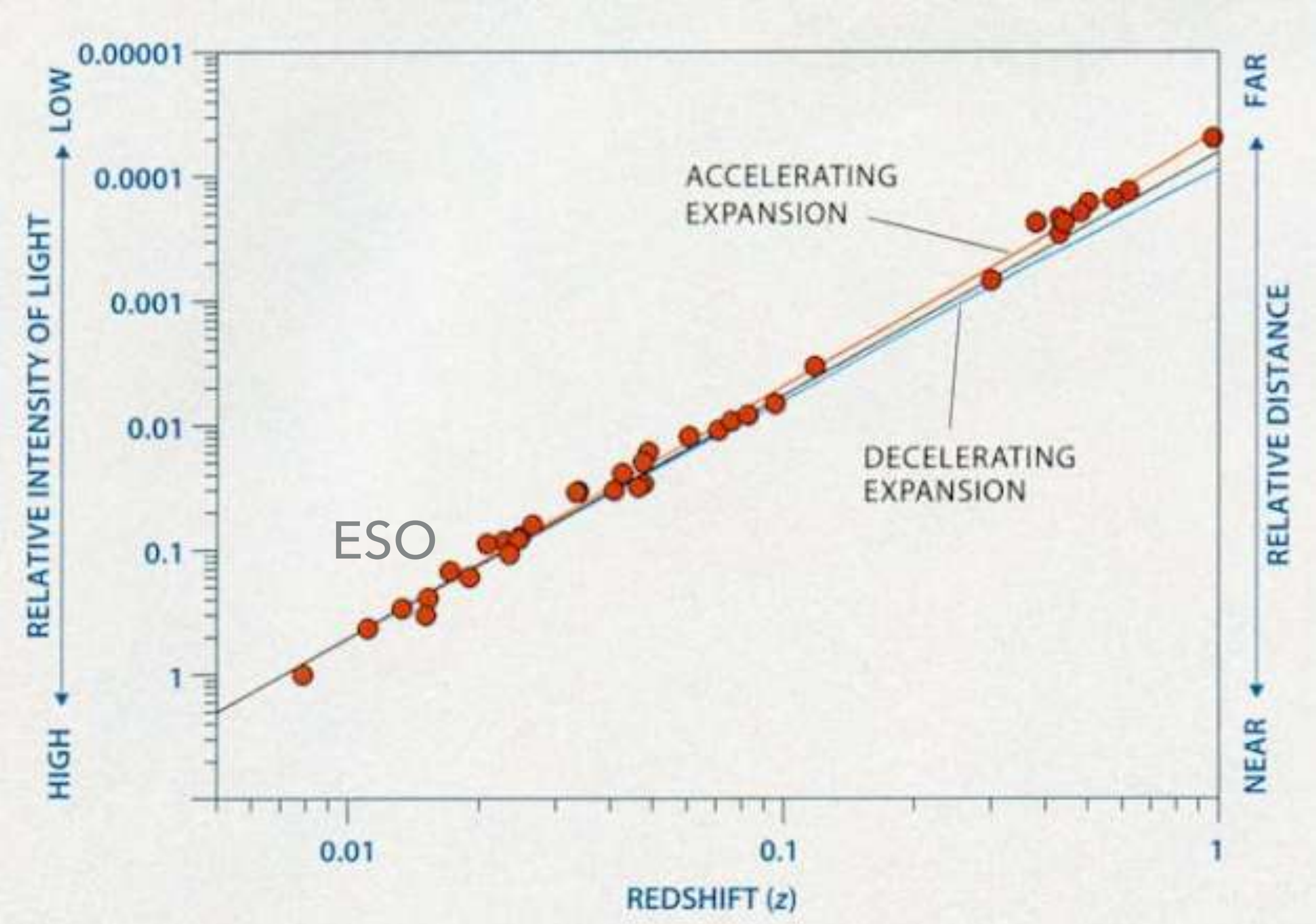


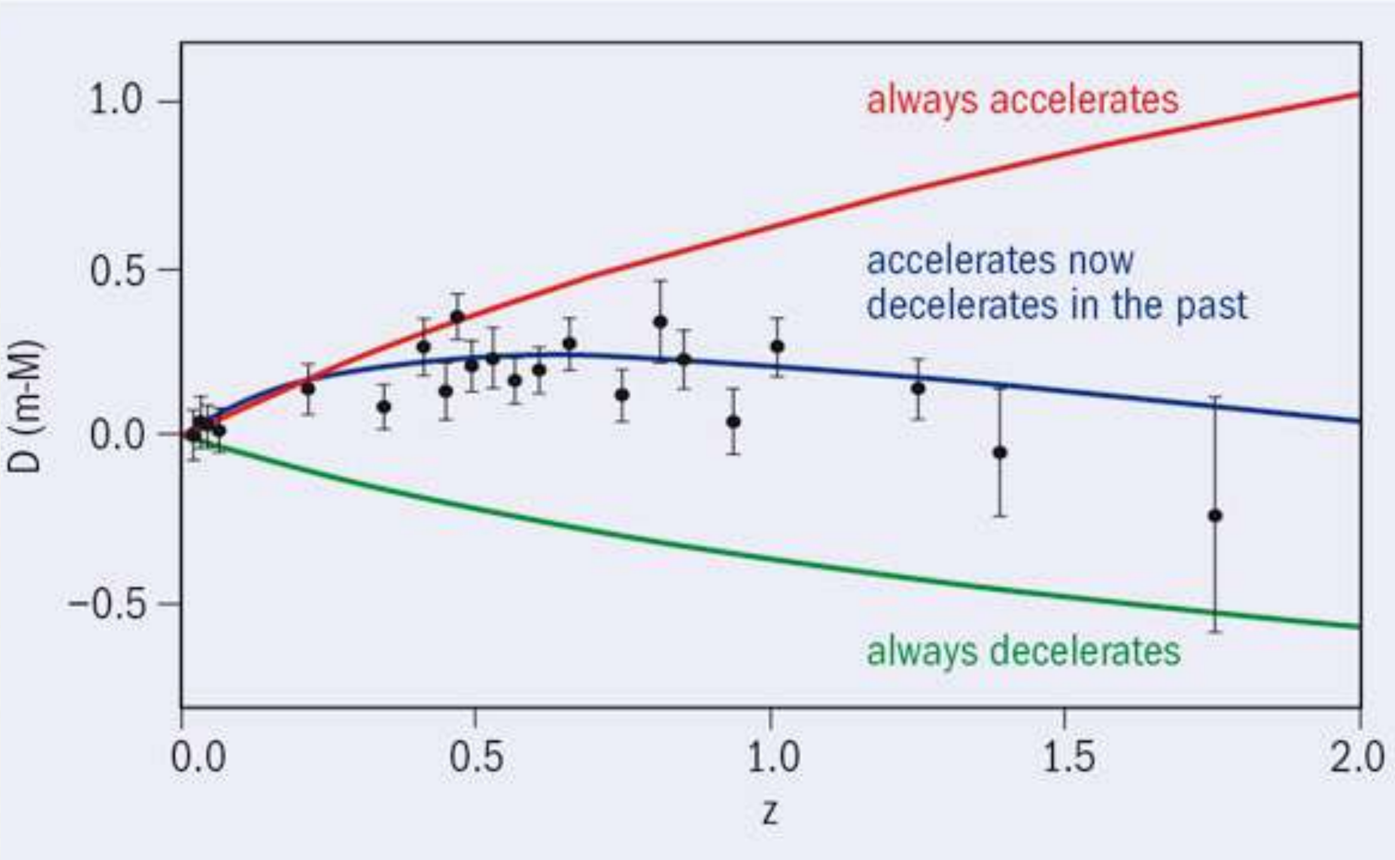
# The Accelerating Expansion of the Universe

The type Ia supernovae
(the standard candle)
$\Rightarrow$ an accelerating expansion

$$cz = H_0 r$$




https://universe-review.ca/

 Credit: Georg Wolschin, University of Heidelberg. http://cerncourier.com/cws/article/cern/37859 

# A "Dark" Component—Dark Energy

- **Dark Energy**, providing a **negative pressure**, is required by the **accelerating expansion**.
- Completely beyond the standard physics
- Dark energy could be just a "**cosmological constant Λ** (Lambda)".
- Understanding dark energy is the top priority in physics.
- So far, only cosmology successfully probes dark energy.

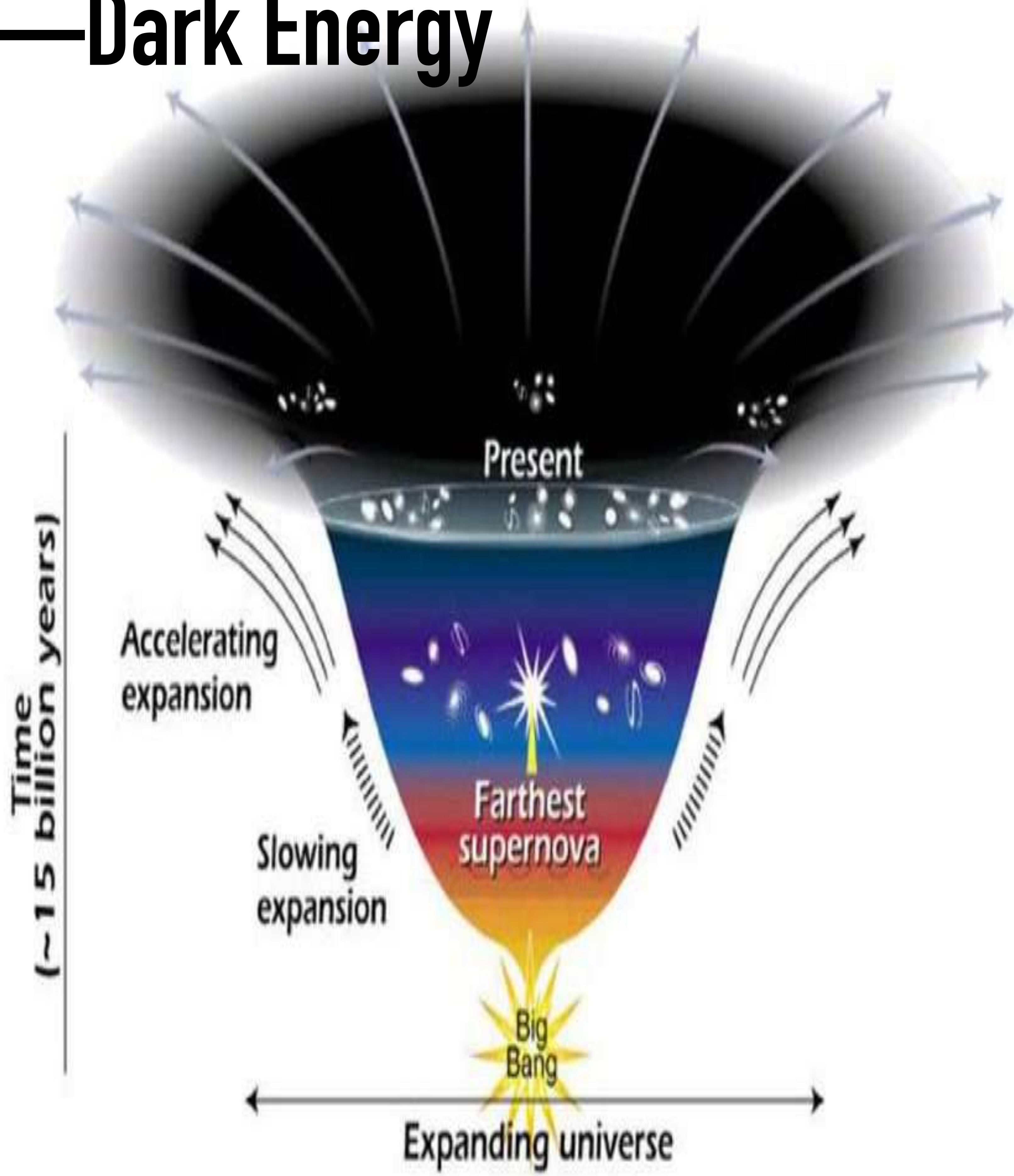
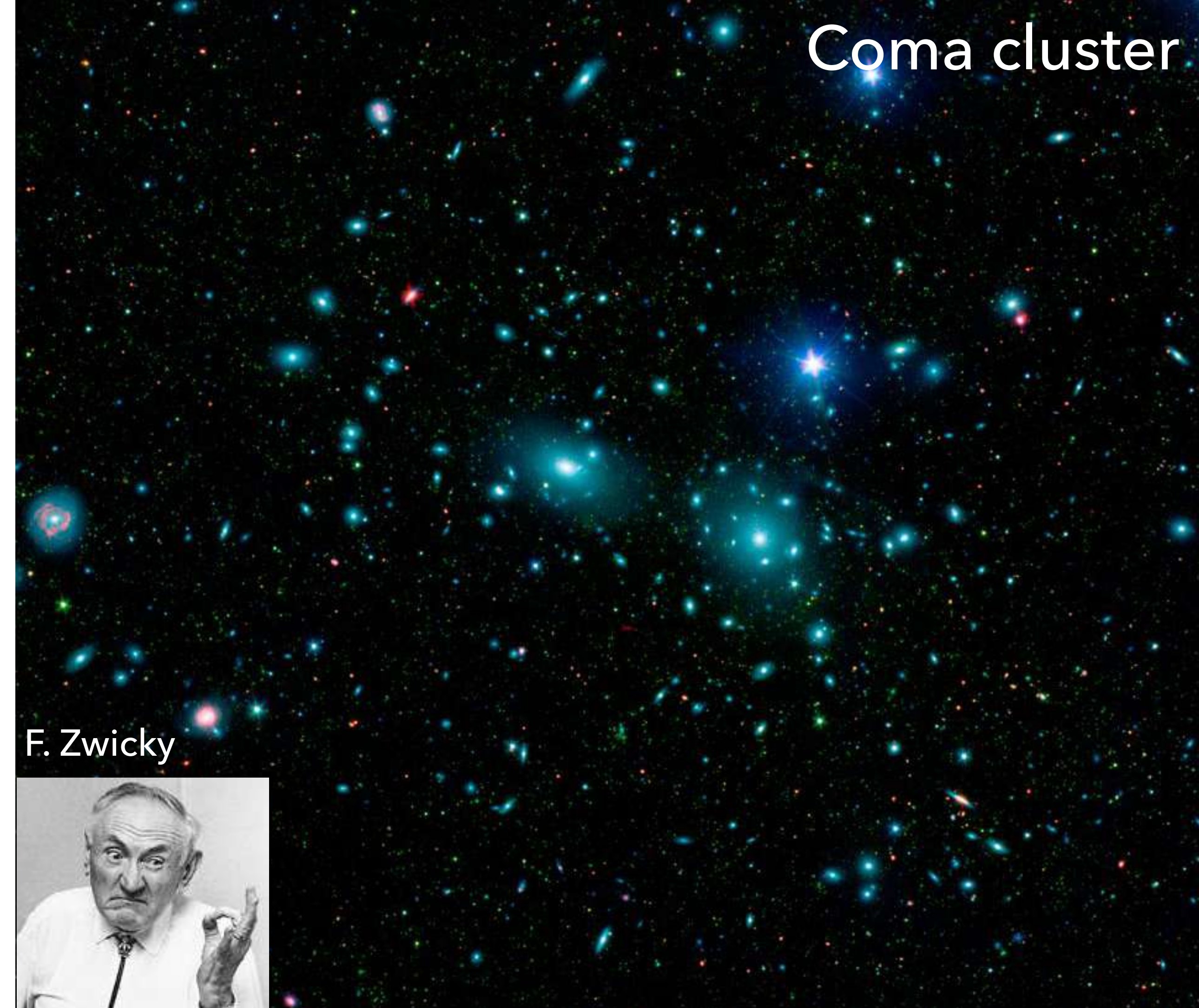




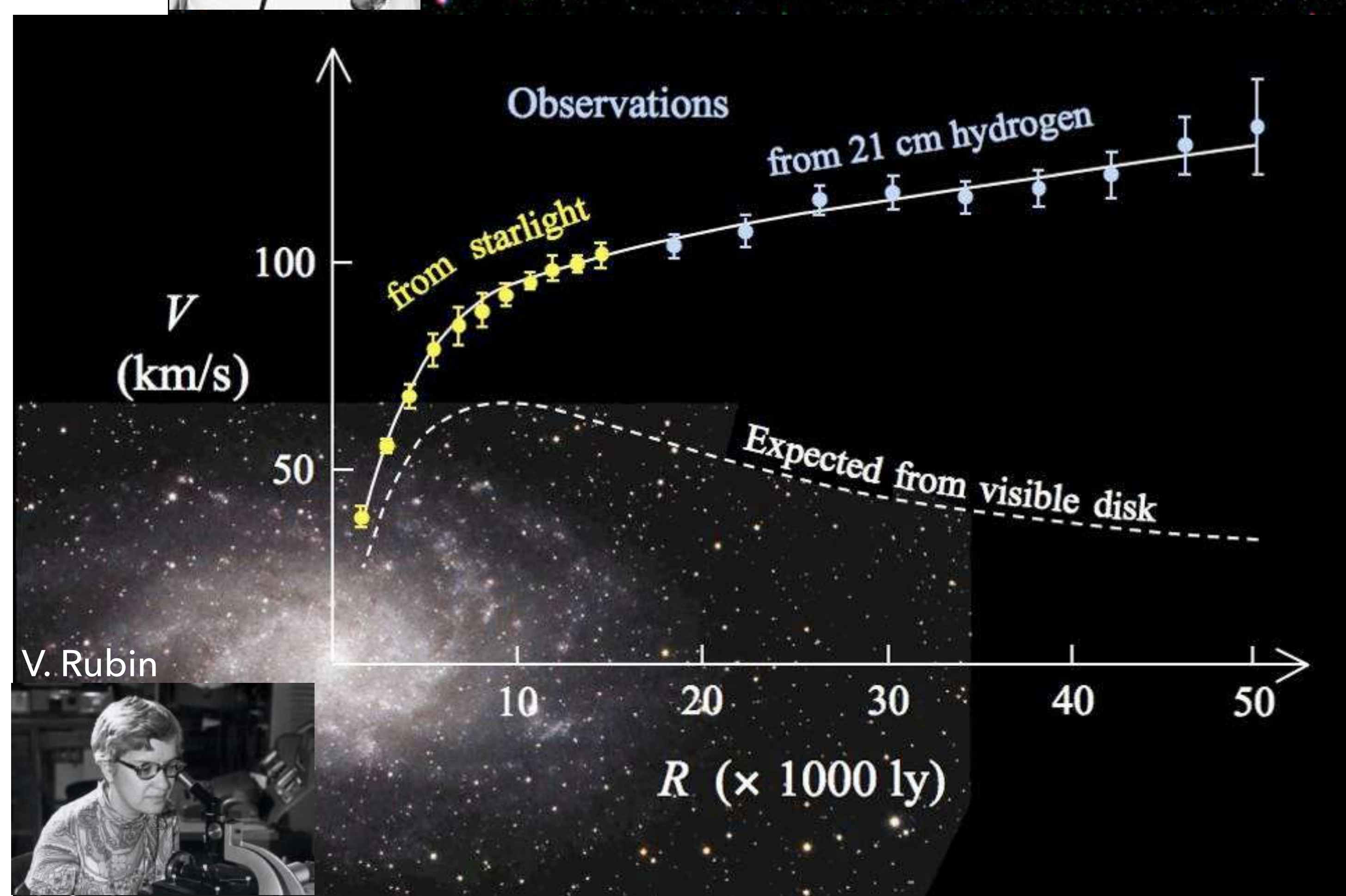

Image courtesy: LBL

# Another Darkness: Dark Matter

- In 1930s, F. Zwicky found that the self-gravity of luminous stars is not enough to support Coma cluster.
  ⇒ *Dunkle Matter* (Dark Matter)
- In 1970s, V. Rubin showed the flat rotation curve.
  ⇒ a kind of unseen matter must exist
- **(Cold) Dark Matter**
  - only interacts via **gravity**
  - comprises a large fraction ($\gtrsim$80%) of matter
  - is beyond the Standard Model
- Dark matter has only been successfully discovered/probed in cosmology.




F. Zwicky




V. Rubin

# The Standard Cosmological Model

- The universe is homogeneous and isotropic at large scales.

- The universe originated from a "big bang" and has been expanding since then.

- The cosmic expansion at the present day is accelerating.

- The **ΛCDM** model: the universe is now composed of
  - ≈5% baryonic matter
  - ≈25% cold dark matter (CDM)
  - ≈70% dark energy (Λ)

- Observational facts supported.


Credit: Discover Magazine

# The Homogeneous Universe

Only little is needed to described the universe:
the **Einstein equations** and the **Boltzmann equation**.

The Einstein equations:
The geometry of spacetime is related to the energy content of the universe.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Che-Yu's lecture

Friedmann–Lemaître–Robertson–Walker (FLRW) metric: **a homogeneous and isotropic expansion**

$$\begin{aligned} ds^2 &= c^2 dt^2 - a(t)^2 \left( \frac{d\chi^2}{1-k\chi^2} + \chi^2 \left( d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \\ &\overset{k=0}{\rightarrow} c^2 dt^2 - \left( dx^2 + dy^2 + dz^2 \right) \end{aligned}$$

# Friedmann equations:

A. Friedmann

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \end{aligned}$$

Recall $H \equiv \frac{\dot{a}}{a}$

$k$ : curvature
$\rho$ : energy density
$p$ : pressure
$\Lambda$ : cosmological constant

Let's make the following assumptions and definitions.

$$
\begin{aligned}
k &\approx 0 \text{ (i.e., flat universe)} \\
H &\equiv \frac{\dot{a}}{a} \text{ (i.e., Hubble Law)} \\
\rho_\Lambda &\equiv \frac{\Lambda c^2}{8\pi G} \text{ (i.e., the energy density of } \Lambda) \\
\rho &= \rho_{\rm m} + \rho_\gamma + \rho_\Lambda \text{ (i.e., assuming only matter, radiations, and } \Lambda) \\
p_i &= w_i \rho_i c^2 \text{ (i.e., the equation of state)}
\end{aligned}
$$

**Friedmann equations:**

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\rho_{\rm m} + \rho_\gamma + \rho_\Lambda\right) \\
H^2 + \dot{H} = \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\sum_{i\in\{{\rm m},\gamma,\Lambda\}}\left[\rho_i\left(1+3w_i\right)\right]
\end{aligned}
$$

The expansion is determined by the content of the universe.

# The First Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\rho_{\rm m} + \rho_\gamma + \rho_\Lambda\right)$$

Cosmic expansion (Hubble parameter) ⇔ Energy density

$$\dot{H} = -4\pi G \sum_{i\in\{{\rm m},\gamma,\Lambda\}} \left[\rho_i \left(1 + w_i\right)\right]$$

Combination of two Friedmann equations.

$$2H\dot{H} = \frac{8\pi G}{3} \sum_{i\in\{{\rm m},\gamma,\Lambda\}} \dot{\rho}_i$$

Time derivative of the first Friedmann equation

$$\begin{aligned}
\sum_{i\in\{{\rm m},\gamma,\Lambda\}} \dot{\rho}_i &= -3H \sum_{i\in\{{\rm m},\gamma,\Lambda\}} \left[\rho_i \left(1 + w_i\right)\right] \\
\dot{\rho}_{\rm m} &= -3H\rho_{\rm m}\left(1 + w_{\rm m}\right) \\
\dot{\rho}_\gamma &= -3H\rho_\gamma\left(1 + w_\gamma\right) \\
\dot{\rho}_\Lambda &= -3H\rho_\Lambda\left(1 + w_\Lambda\right)
\end{aligned}$$

Density evolutions depend on the Hubble parameter.

Recall $H \equiv \frac{\dot{a}}{a}$

We know $w_{\mathrm{m}} = 0$ and $w_\gamma = \frac{1}{3}$ :

$$\frac{\mathrm{d}\rho_{\mathrm{m}}}{\rho_{\mathrm{m}}} = -3\frac{\mathrm{d}a}{a} \Rightarrow \rho_{\mathrm{m}} \propto a^{-3}$$

$$\frac{\mathrm{d}\rho_\gamma}{\rho_\gamma} = -4\frac{\mathrm{d}a}{a} \Rightarrow \rho_\gamma \propto a^{-4}$$

$$\frac{\mathrm{d}\rho_\Lambda}{\rho_\Lambda} = -3\left(1 + w_\Lambda\right)\frac{\mathrm{d}a}{a} \Rightarrow \rho_\Lambda \propto a^{-3(1+w_\Lambda)}$$

The density of radiations decays faster than that of matter.

Energy densities depend on a and w

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{m},0}a^{-3} + \rho_{\gamma,0}a^{-4} + \rho_{\Lambda,0}a^{-3(1+w_\Lambda)}\right)$$

$$\boxed{H^2 = {H_0}^2\left(\Omega_{\mathrm{m}}a^{-3} + \Omega_\gamma a^{-4} + \Omega_\Lambda a^{-3(1+w_\Lambda)}\right)}$$

The Hubble parameter scaling depends on energy fractions

$$\rho_{\mathrm{crit},0} \equiv \frac{3H_0^2}{8\pi G}$$

Current critical density

$$\Omega_{\mathrm{m}} \equiv \frac{\rho_{\mathrm{m},0}}{\rho_{\mathrm{crit},0}}$$

Current energy fraction of matter

$$\Omega_\gamma \equiv \frac{\rho_{\gamma,0}}{\rho_{\mathrm{crit},0}}$$

Current energy fraction of radiations

$$\Omega_\Lambda \equiv \frac{\rho_{\Lambda,0}}{\rho_{\mathrm{crit},0}}$$

Current energy fraction of “dark energy”

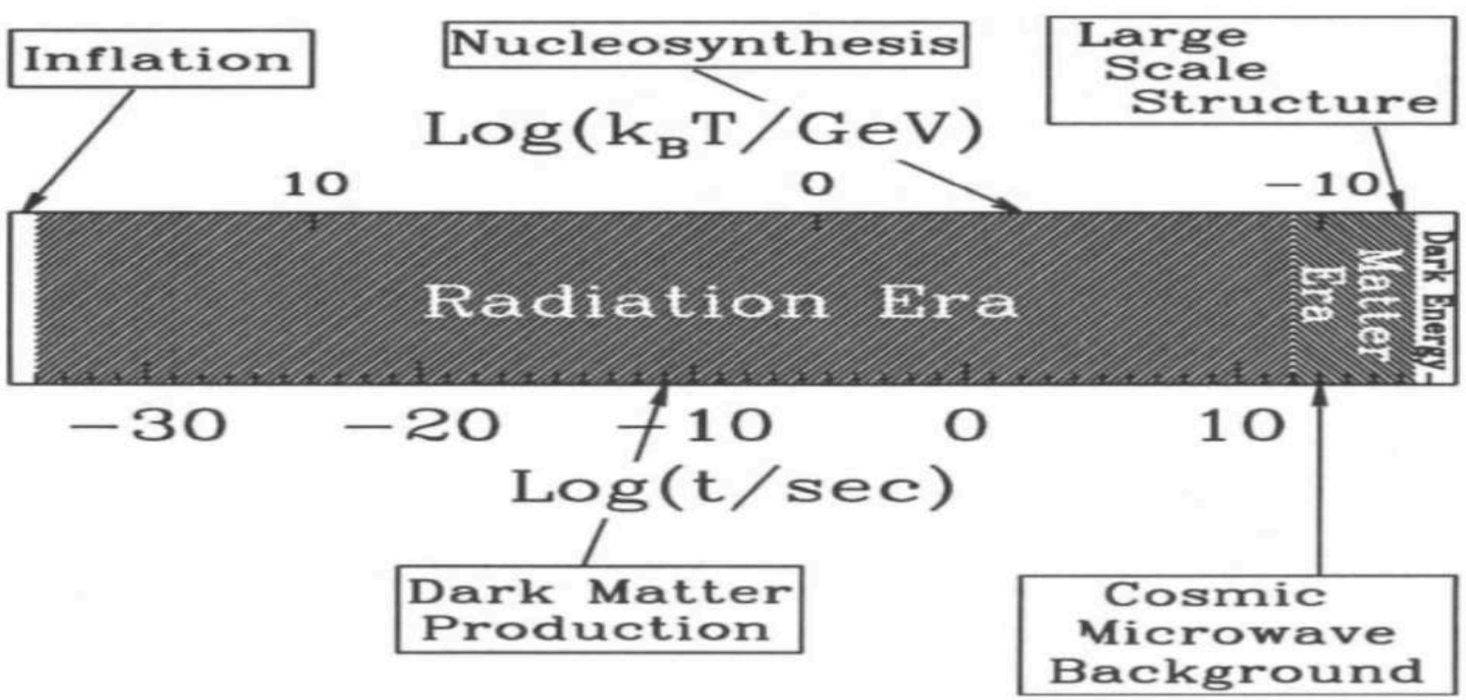
The first Friedmann equation in a general form:

$$H^2 = {H_0}^2 \left( \Omega_\gamma a^{-4} + \Omega_{\mathrm{m}} a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda a^{-3(1+w_\Lambda)} \right)$$

At the present day, we observe $\Omega_{\mathrm{m}} \approx 0.3, \Omega_\gamma \approx 10^{-4}, \Omega_\Lambda \approx 0.7,$

Based on the scaling, there was a time at $a \approx 3\times10^{-4}$ or $z \approx 3000$ when the universe is in the matter-radiation equality.

$$\frac{\rho_{\mathrm{m}}}{\rho_\gamma} = \frac{\rho_{\mathrm{m},0}\, a^{-3}}{\rho_{\gamma,0}\, a^{-4}} \approx \frac{0.3}{10^{-4}} a \approx 3000a \approx 3000 \times \frac{1}{1+z}$$




S. Dodelson
Modern Cosmology

$$\begin{aligned}
\text{radiation} - \text{matter equality}: &\quad a_{\gamma\mathrm{m}} \approx 10^{-4} \quad z_{\gamma\mathrm{m}} \approx 3440 \quad t_{\gamma\mathrm{m}} \approx 50{,}000 \text{ yr} \\
\text{matter} - \Lambda \text{ equality}: &\quad a_{\mathrm{m}\Lambda} \approx 0.77 \quad z_{\mathrm{m}\Lambda} \approx 0.3 \quad t_{\mathrm{m}\Lambda} \approx 10 \text{ Gyr} \\
\text{now}: &\quad a = 1 \quad z = 0 \quad t \approx 14 \text{ Gyr}
\end{aligned}$$

# The Second Friedmann (Acceleration) Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\left(1+3w\right)$$

$\boxed{\text{The ac/deceleration of the expansion} \Leftrightarrow \text{the nature of energy contents}}$

$$\text{Deceleration} \quad \Leftrightarrow \quad \ddot{a} < 0 \quad \Leftrightarrow \quad w > -\tfrac{1}{3}$$

$$\text{Acceleration} \quad \Leftrightarrow \quad \ddot{a} > 0 \quad \Leftrightarrow \quad w < -\tfrac{1}{3}$$

In the radiation and matter dominated eras, the expansion is decelerating.

The accelerating expansion ⇒ w < -1/3.

What is dark energy?

If w = -1, $\dot{\rho}_\Lambda = -3H\rho_\Lambda\left(1+w\right) = 0$

If w < -1, $\dot{\rho}_\Lambda > 0$

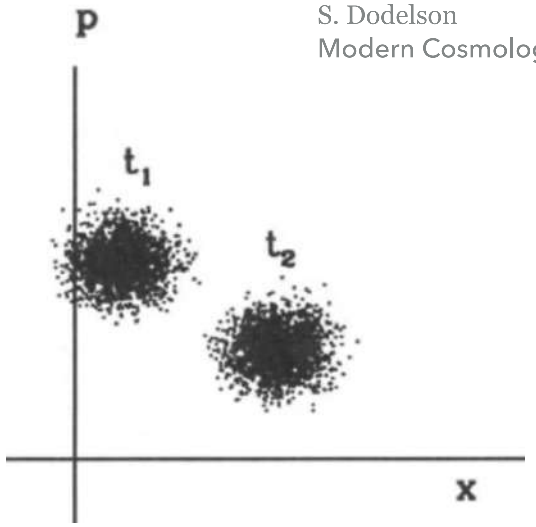
If w > -1, $\dot{\rho}_\Lambda < 0$

The cosmological constant (Λ)

So far, all observations imply w ≈ -1.

# The Inhomogeneous Universe

Imagine all species (matter, radiations, etc) as fluids of cosmological particles with velocities which evolve in time. The goal is to solve the distribution $f(\vec{x}, \vec{p}, t)$. This distribution is described by the **Boltzmann equation**. $\frac{\mathrm{d}f}{\mathrm{d}t} = C[f]$



S. Dodelson
Modern Cosmology

Consider a simple case: non-relativistic particles without collisions with other species in a universe with a homogeneous and isotropic expansion (FLRW metric). The Boltzmann equation leads to the continuity equation and the Euler equation.

The continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$

The Euler equation: $\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\Phi$

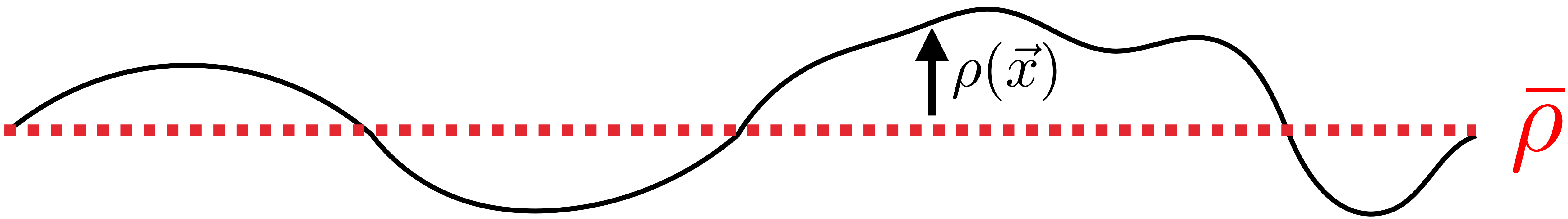
The Poisson equation: $\nabla^2 \Phi = 4\pi G \rho$

Solvable

The zero-order solution: the uniform background $\rho(\vec{x}, t) = \bar{\rho}(t)$ and the Hubble flow $\vec{v} = H\vec{x}$.

What we care about is the "**first-order**" perturbation.

# Linear Perturbations




The structure is a (linear) perturbation to the background. We can rewrite:

$$\begin{aligned} \rho &= \bar{\rho} + \delta\rho \\ \vec{v} &= \bar{\vec{v}} + \delta\vec{v} \\ p &= \bar{p} + \delta p \\ \Phi &= \bar{\Phi} + \delta\Phi \end{aligned}$$

The zero-order quantities describe the background (Friedmann equations).

The first-order quantities describe linear structures.

Collecting the first-order terms:

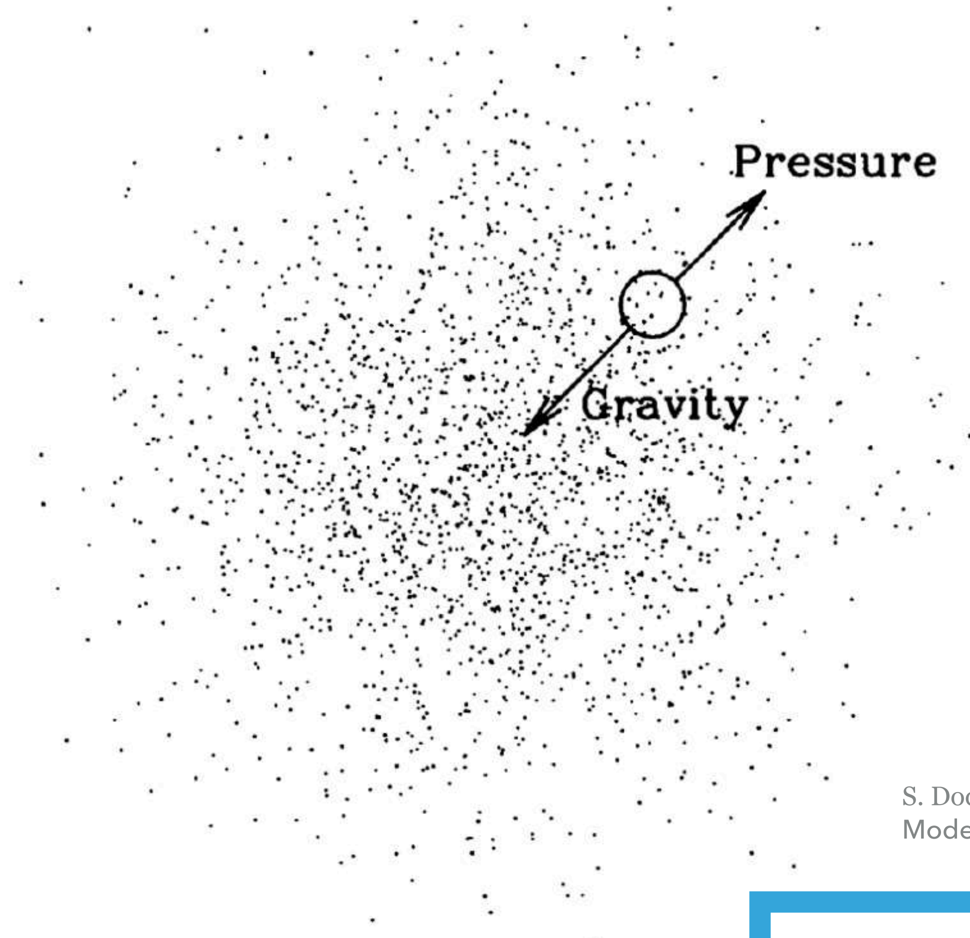
$$\ddot{\delta} + 2H\dot{\delta} = \left(4\pi G\bar{\rho} - c_{\mathrm{s}}^2\nabla^2\right)\delta$$

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \qquad c_{\mathrm{s}}^2 \equiv \frac{\delta p}{\delta\rho}$$

The Hubble flow is the damping term.

$\delta$: The overdensity of perturbations

# The Growth of Structures in a Static Universe




S. Dodelson
Modern Cosmology

Consider H = 0:

$$\ddot{\delta} = \left(4\pi G\bar{\rho} - c_{\rm s}^2\nabla^2\right)\delta$$

$$\lambda_{\rm J} \equiv c_{\rm s}\sqrt{\frac{\pi}{G\bar{\rho}}}$$

$$\lambda > \lambda_{\rm J} \quad \Rightarrow \quad \text{Gravitational collapse.}$$

$$\lambda < \lambda_{\rm J} \quad \Rightarrow \quad \text{Pressure dominated.}$$

There are two competing forces, the self-gravity and the pressure. This leads to the so-called "**Jeans instability**".

Structures grow exponentially in a static universe.

Perturbations propagate as sound waves.

# The Growth of CDM Structures

Let's assume that the particles are cold dark matter (pressureless):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0$$

$$t_{\rm H} \propto \frac{1}{H} \propto \frac{1}{\sqrt{\rho}} \qquad t_{\rm collapse} \propto \frac{1}{\sqrt{\rho_{\rm m}}}$$

$t_{\rm H} \lesssim t_{\rm collapse} (i.e., \rho \gtrsim \rho_{\rm m})$: The perturbation grows very slowly. That is, structures form slowly if the cosmic density is *not* dominated by matter.

$t_{\rm H} \approx t_{\rm collapse} (i.e., \rho \approx \sqrt{\rho_{\rm m}})$: The perturbation grows. Moreover, structures grow as a power law of time (not exponentially).

One can easily show that the solution to the perturbation equation is "slowed" if the the Hubble expansion is a constant.

**The perturbation equation**

The competition between the expansion and the self-gravity.

In the radiation-dominated era, the structures do not grow significantly (the **Meszaros effect**).

In the matter-dominated era, the structures grow "linearly". **The linear growth**: $\delta(\vec{x},t) = \delta(\vec{x})D(t) \propto \delta(\vec{x})t^{\frac{2}{3}} \propto \delta(\vec{x})a(t)$

In the "cosmological constant ($\Lambda$)" dominated era, the structures remain constant or decays.

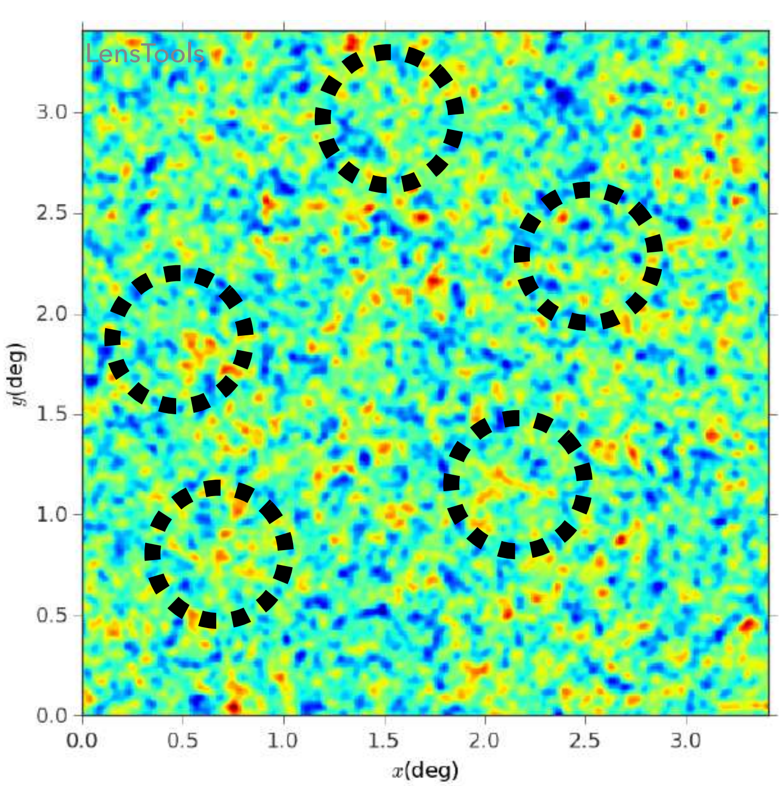
# The Density Field—the Gaussian Random Field

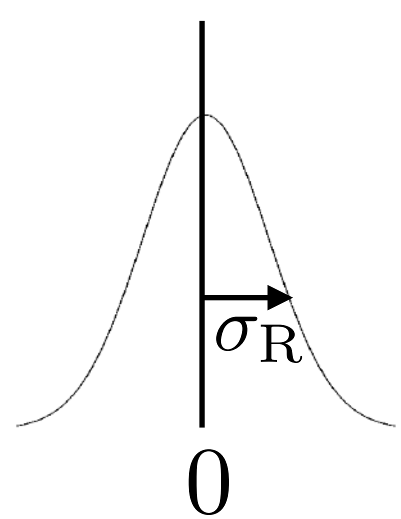
So far, we know how the density perturbations evolve in time (asymptotically). To solve the full perturbation equation, we need is the initial perturbations $\delta(\vec{x}, t=0)$.

Right after the big bang, the universe experienced an extremely rapid (60 e-fold) expansion that we call the "**inflation**". The "**quantum fluctuation**" during the inflation becomes the "**primordial fluctuation**", as the initial perturbation.

A general (and quite intuitive!) assumption is that the primordial perturbation is a **Gaussian random field**. Specifically, given a scale of interested, the smoothed overdensity field can be solely described by a **variance**.

$$\Delta^2_{\mathrm{R}}(k) \equiv \frac{1}{2\pi^2} k^3 P(k)$$



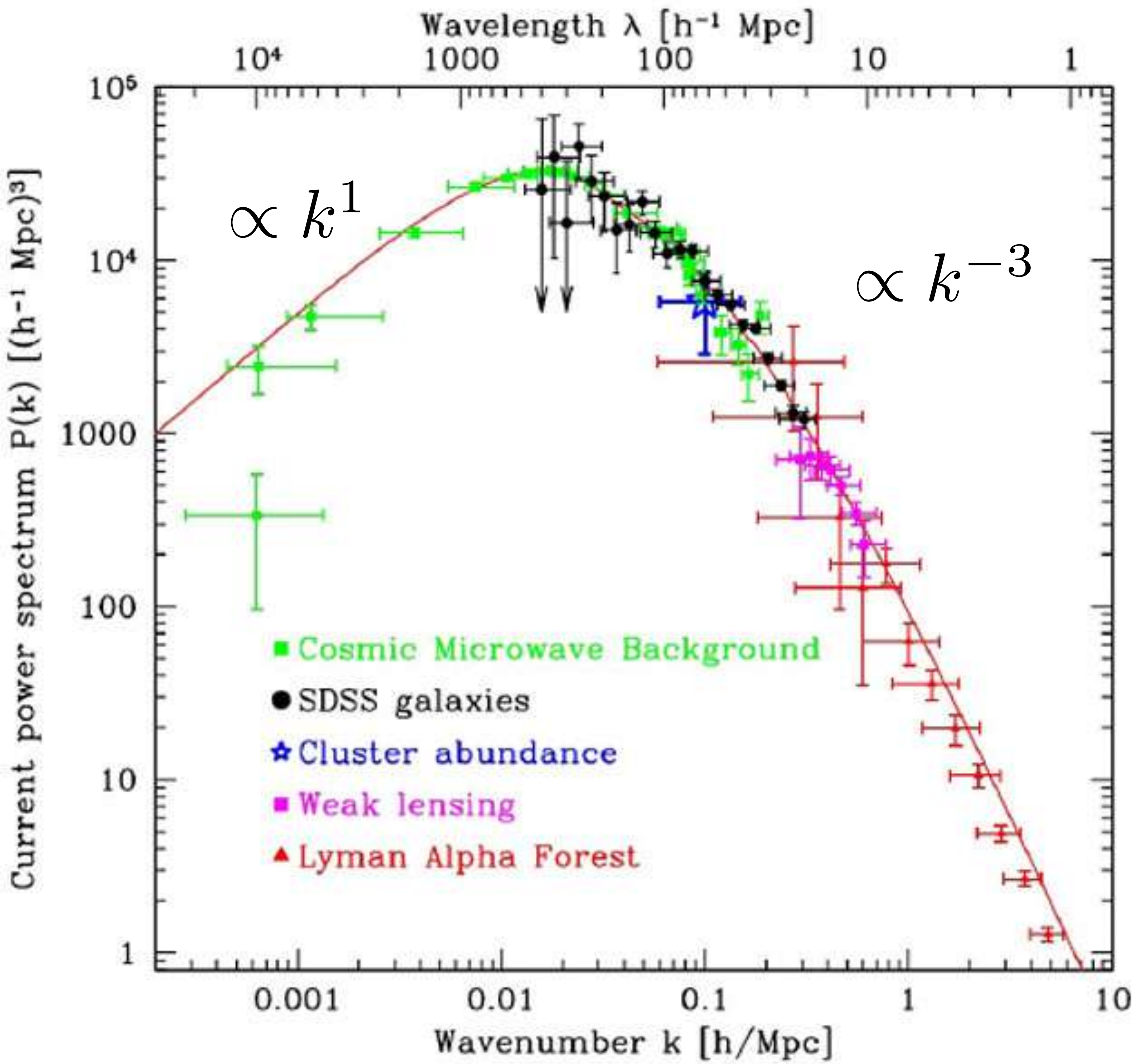

Smaller R ⇔ larger k~ 1/R ⇔ larger variance.

Determining the power spectrum P(k) is big in cosmology.

We know the **initial power spectrum** $P_0(k)$~k^0.97. We know the evolution (**growth function**). Determining the **normalization** is effectively probing the primordial perturbation.

# The Matter Power Spectrum

$$P(k,a) \propto P_{\rm p}(k)\ T^2(k)\ D^2(a)$$

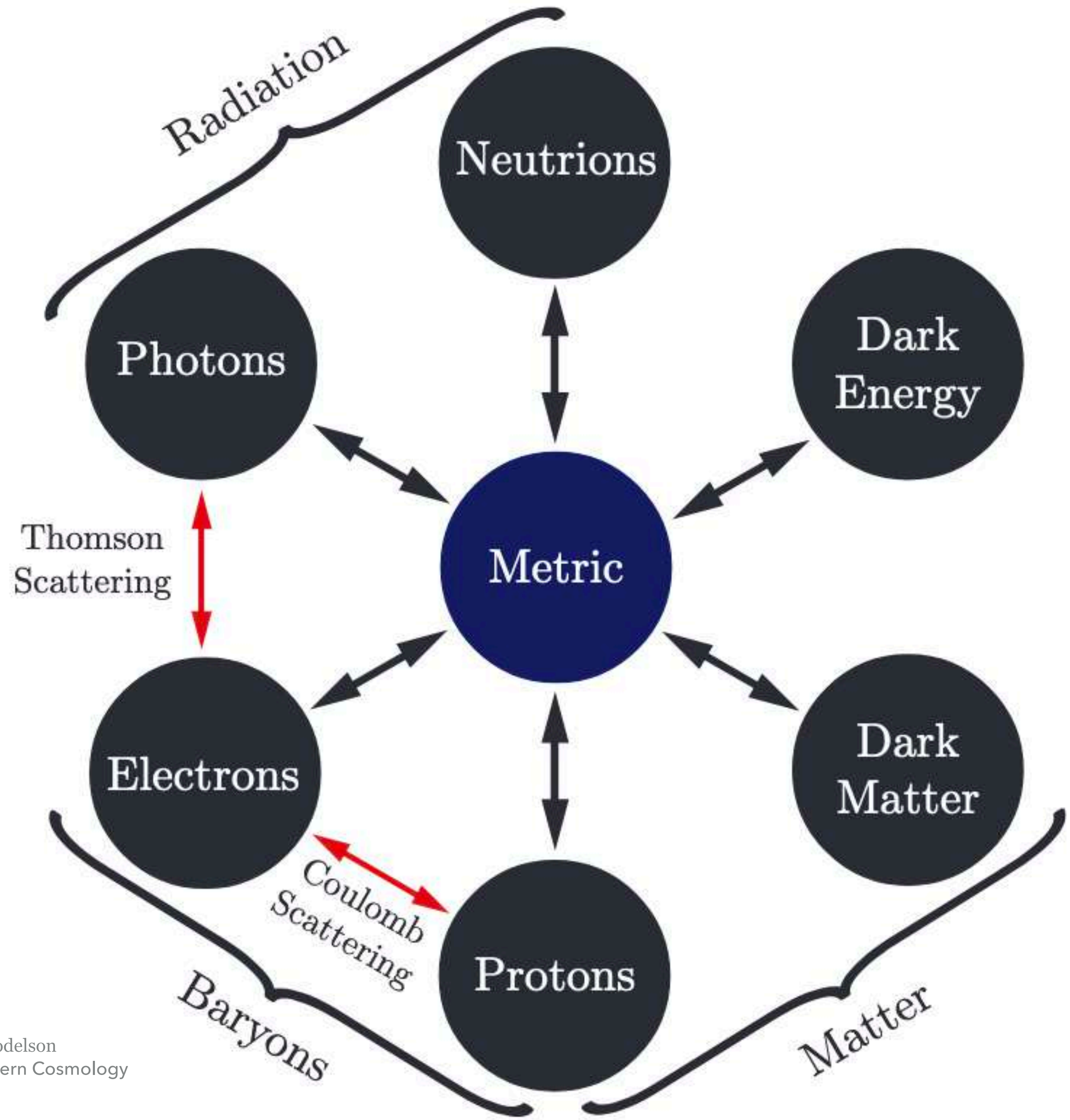



T(k) the transfer function
D(a) the growth factor
$P_p$(k) the primordial spectrum

The matter power spectrum is ∝ k (∝ $k^{-3}$) at large (small) scales.

At large scales, the slope is set up by the primordial spectrum, which is referred to a "**scale-invariant spectrum**" if $P_p$(k) ∝ k.

# The Full Solutions to the Boltzmann Equation

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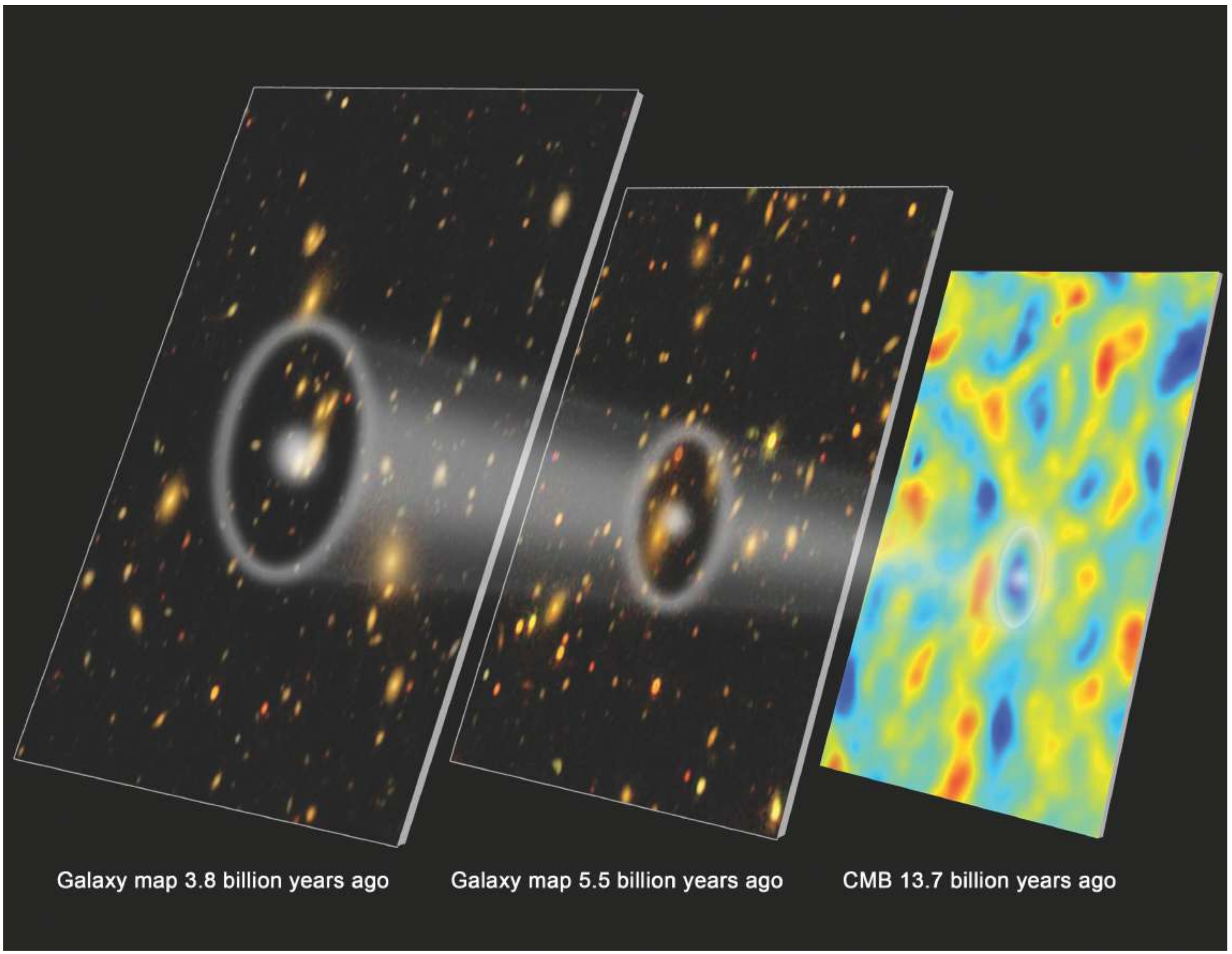
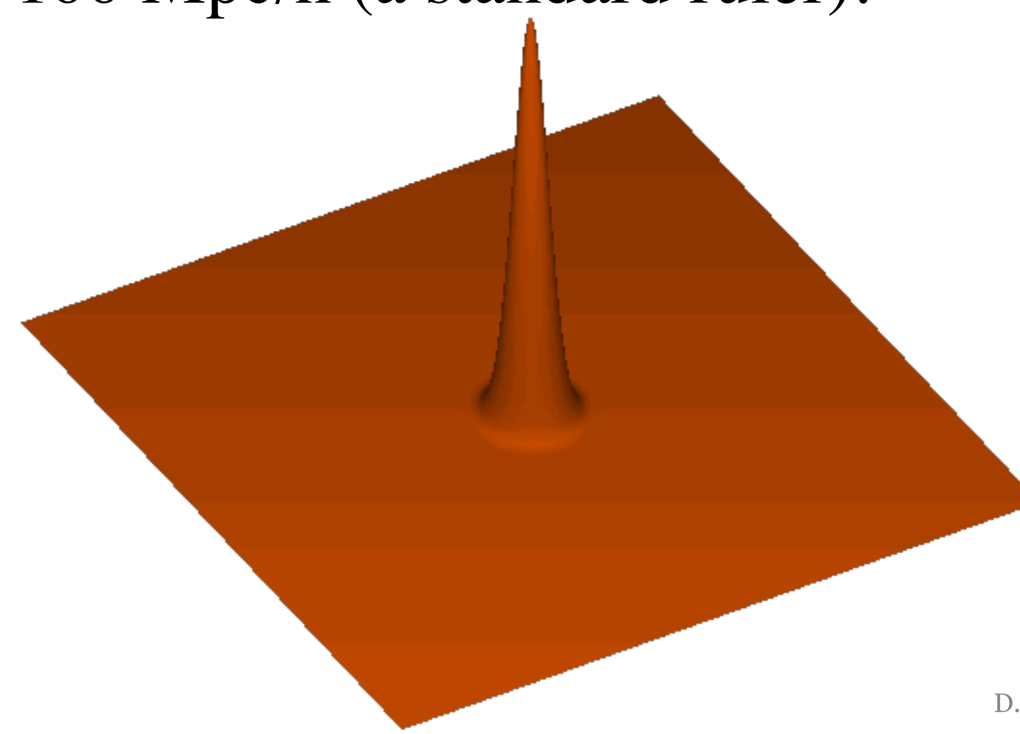
Collision terms $\neq 0$

# Measurements of the Universe

# Baryonic Acoustic Oscillations (BAO)




In the early universe, perturbations propagate as sound waves in the photon-baryon fluid. At the recombination, photons start to move freely. Meanwhile, the perturbations freeze at a fixed scale of ≈100 Mpc/h (a standard ruler).

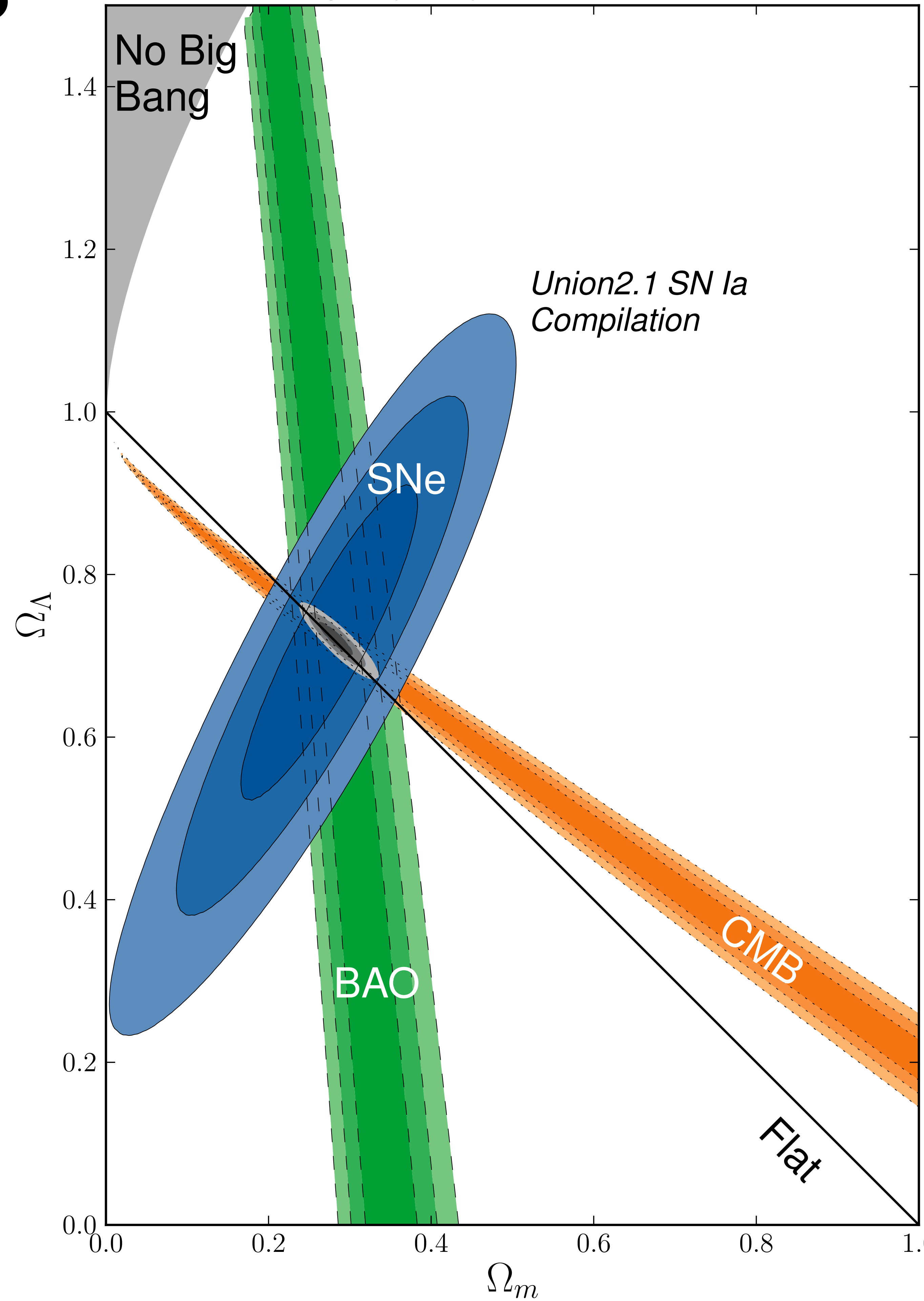


D. Eisenstein

# Cosmological Constraints

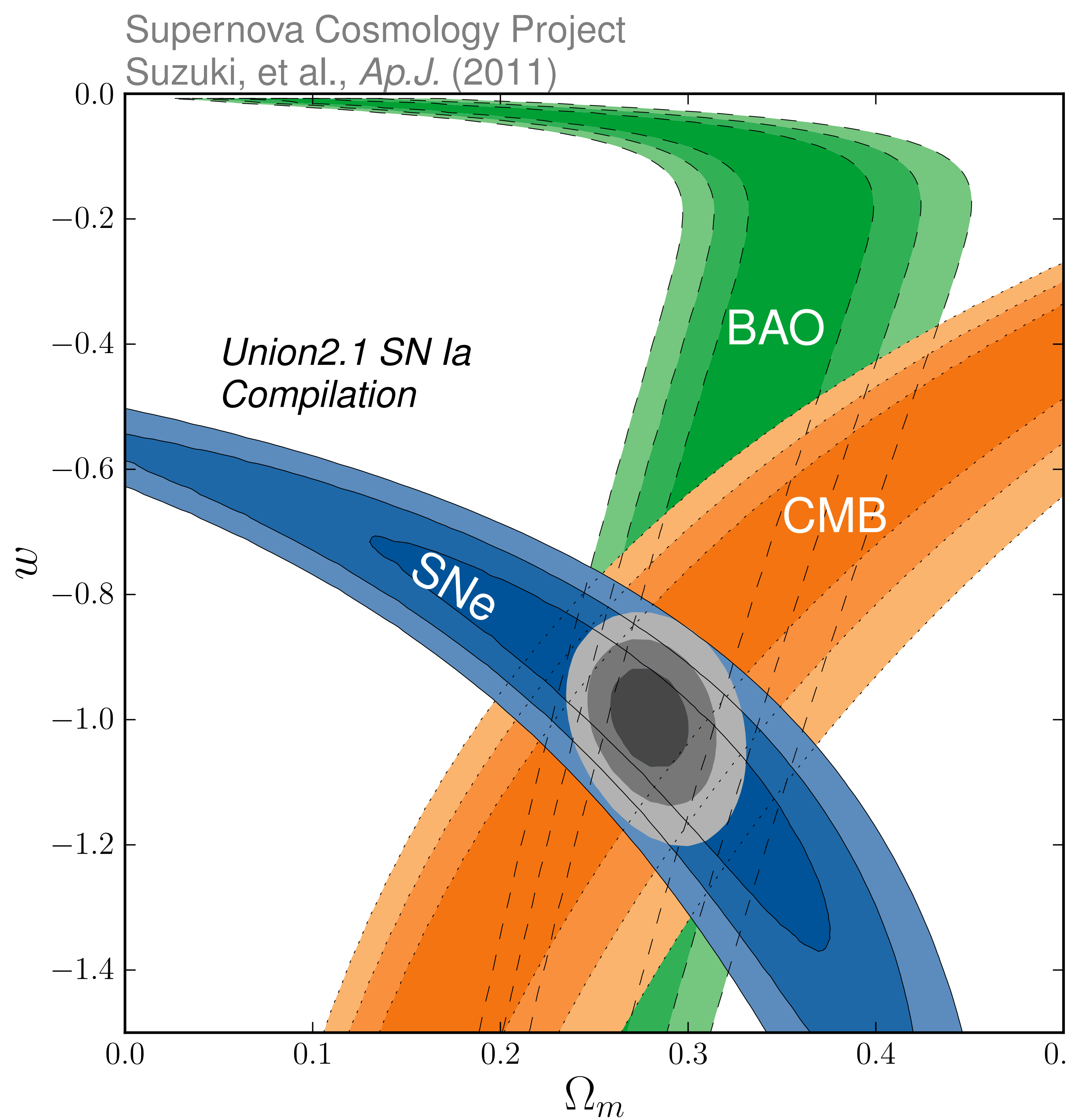
The universe is flat and has $\Omega_{\mathrm{m}} \approx 0.3, \Omega_{\Lambda} \approx 0.7$

Supernova Cosmology Project
Suzuki, et al., *Ap.J.* (2011)

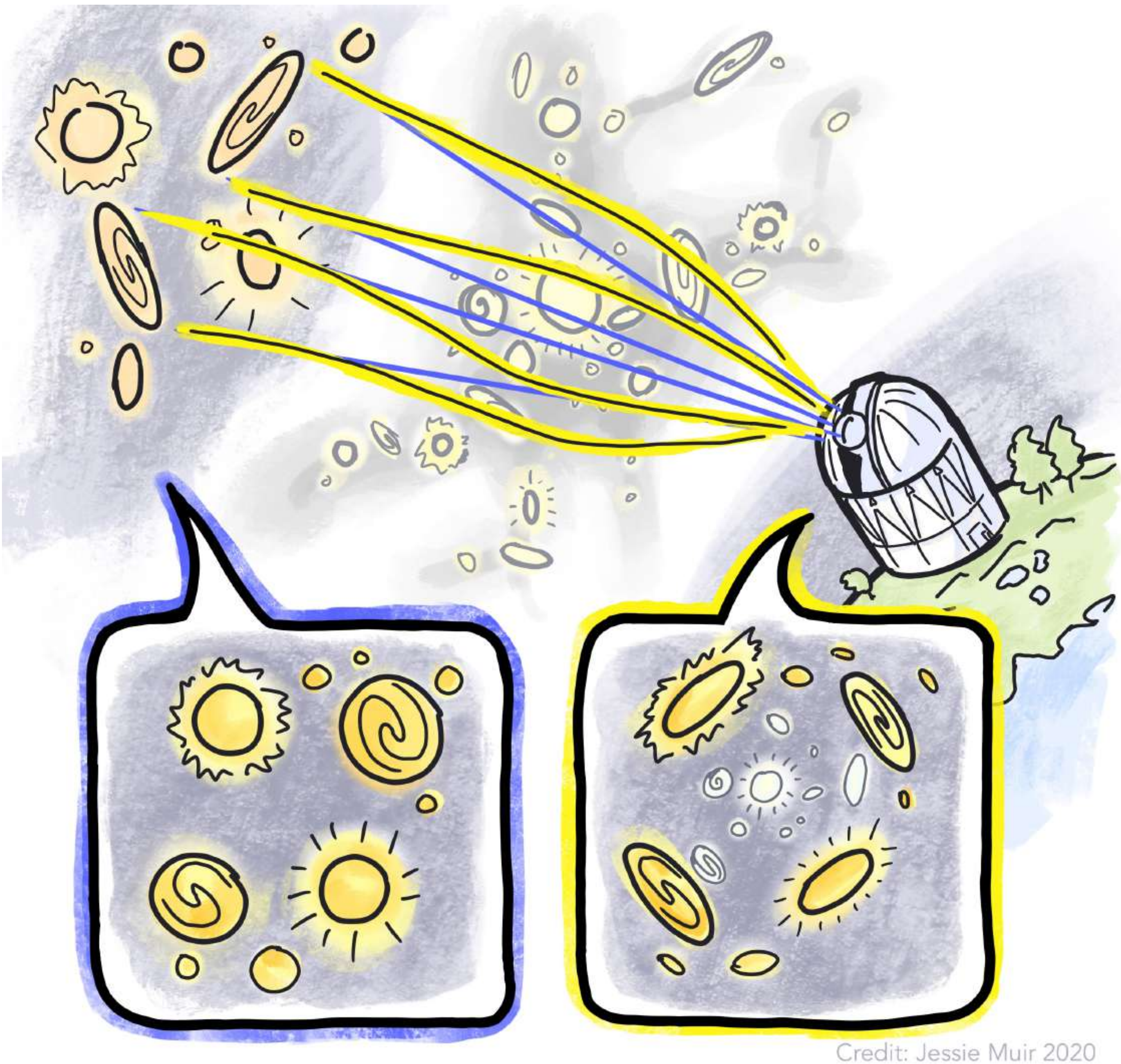


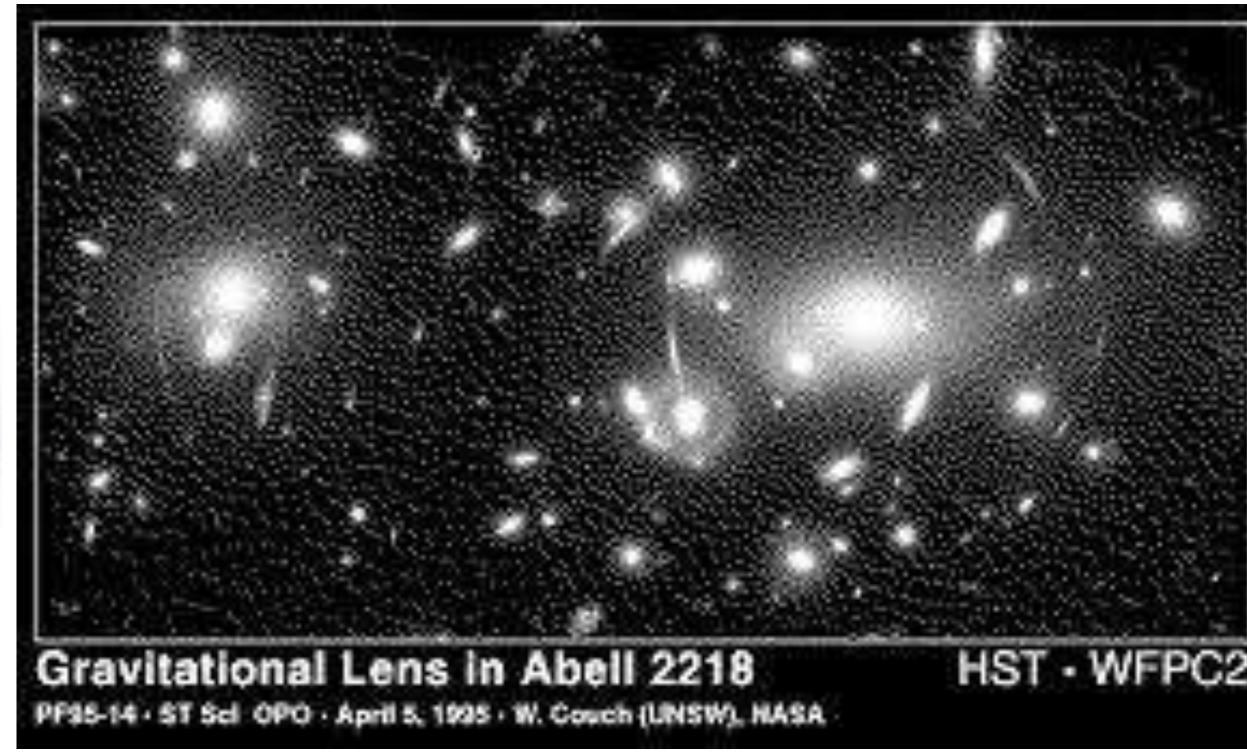
# Cosmological Constraints

All observations are consistent with
$w = -1$



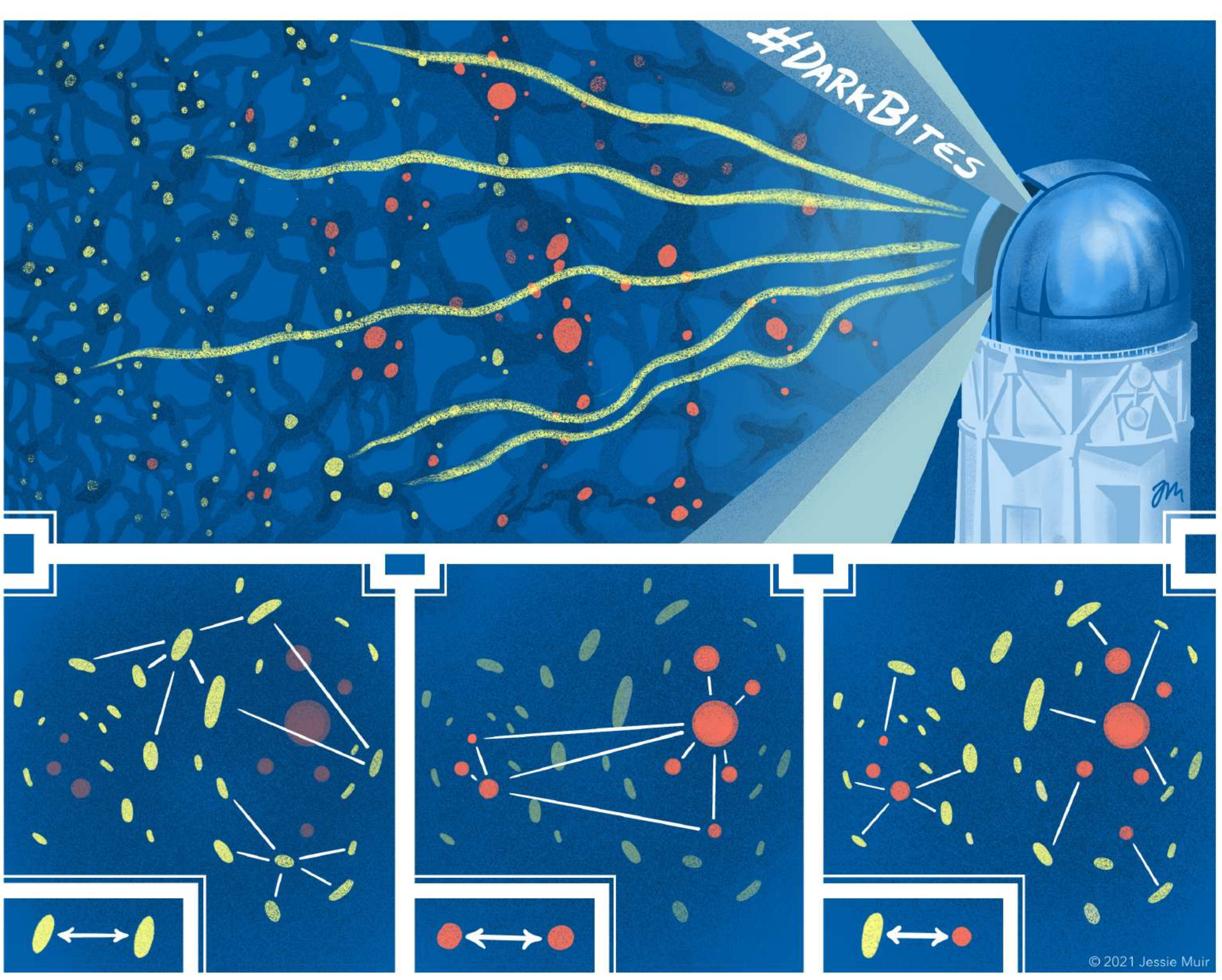
# Weak Gravitational Lensing




The technique of weak gravitational lensing probes the total potential, providing an extremely powerful tool for cosmology.

# Weak Lensing and Clustering of Structures




Lensing x Lensing

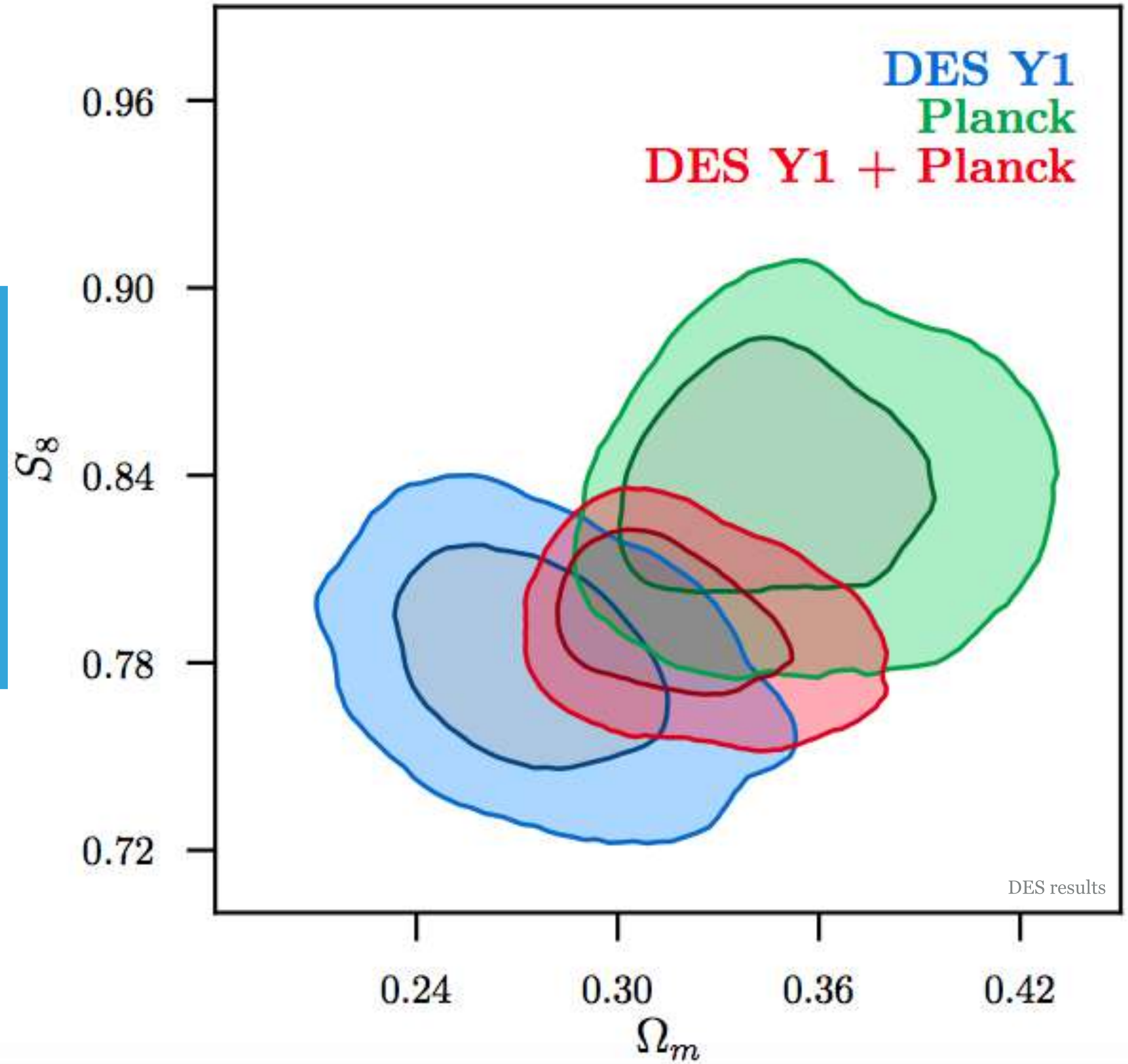
Clustering

Weak lensing

# Cosmological Constraints

$$S_8 \equiv \sigma_8 \sqrt{\Omega_\mathrm{m}/0.3}$$
$$\propto \sigma_8 \equiv \text{overdensity fluctuations at a scale of 8 Mpc/h}$$

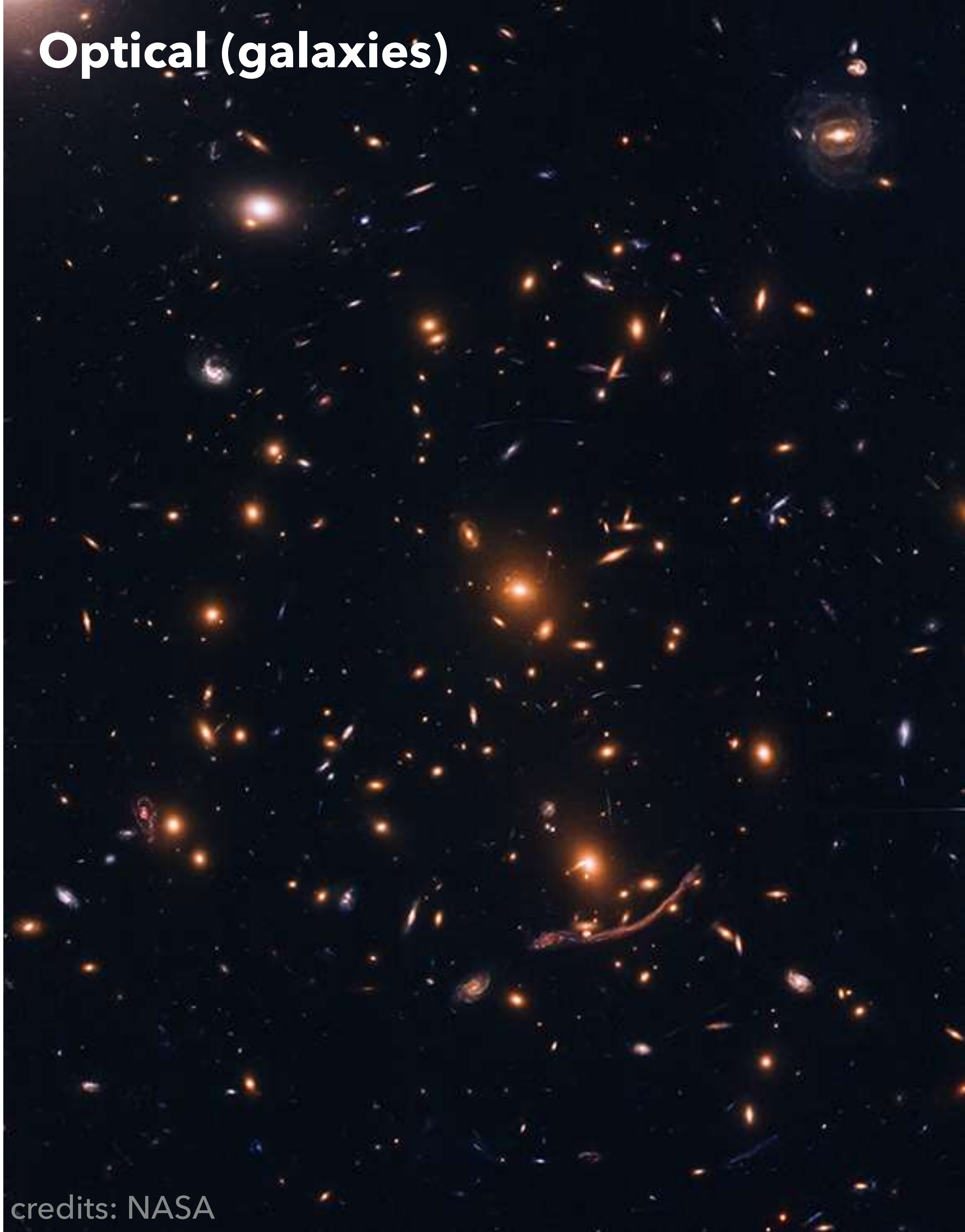
All probes infer a consistent picture of structure formations.



# Galaxy Clusters

- The largest systems in the universe
- Extremely massive ($M \approx 10^{15}$ $M_\odot$) and large ($R \approx$ Mpc)
- $\approx$80% mass in dark matter
- $\approx$20% mass in baryons
  - $\approx$5% mass in stars (galaxies)
  - $\approx$15% mass in hot plasma (intracluster medium, ICM)

An ideal cosmic laboratory!



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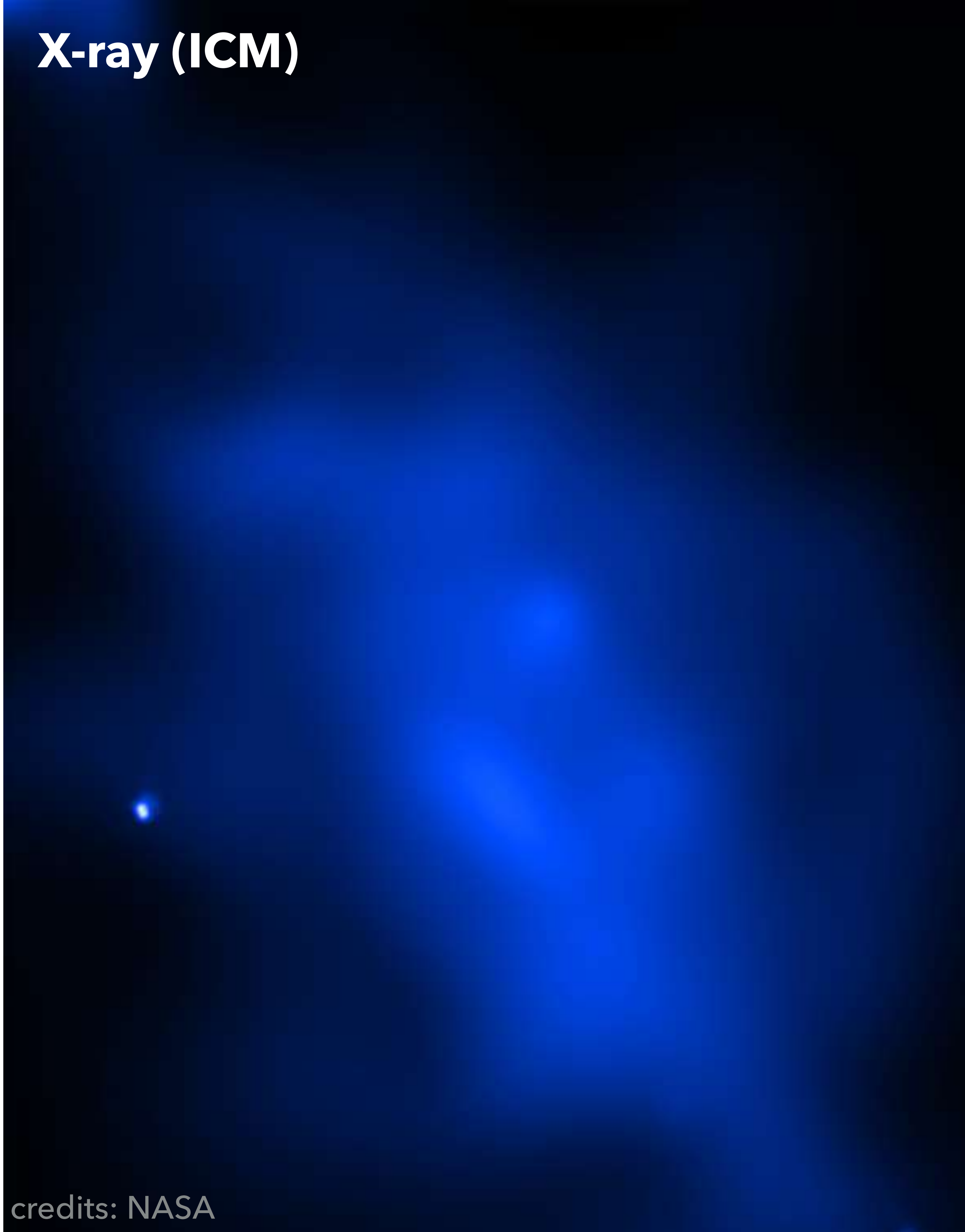

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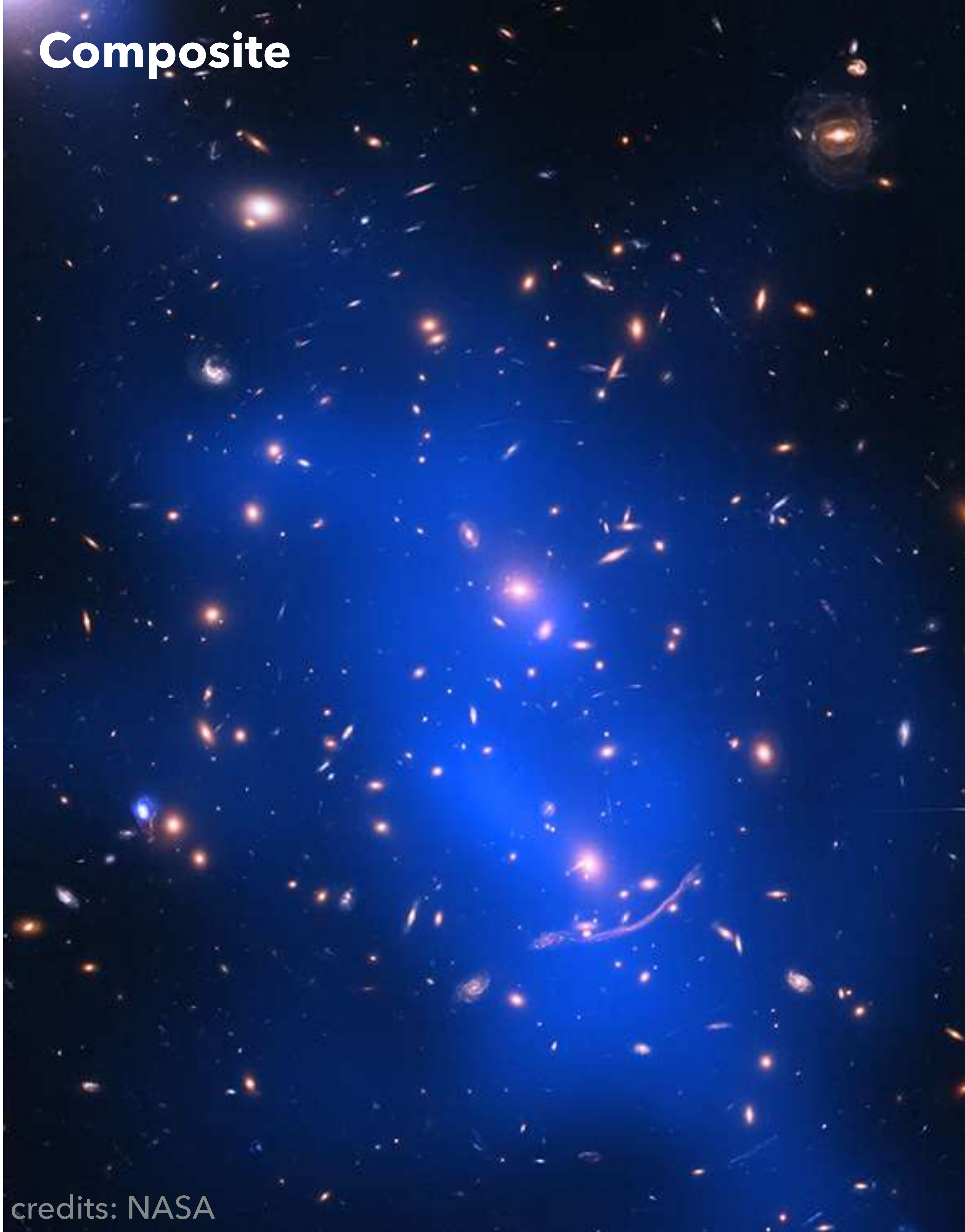
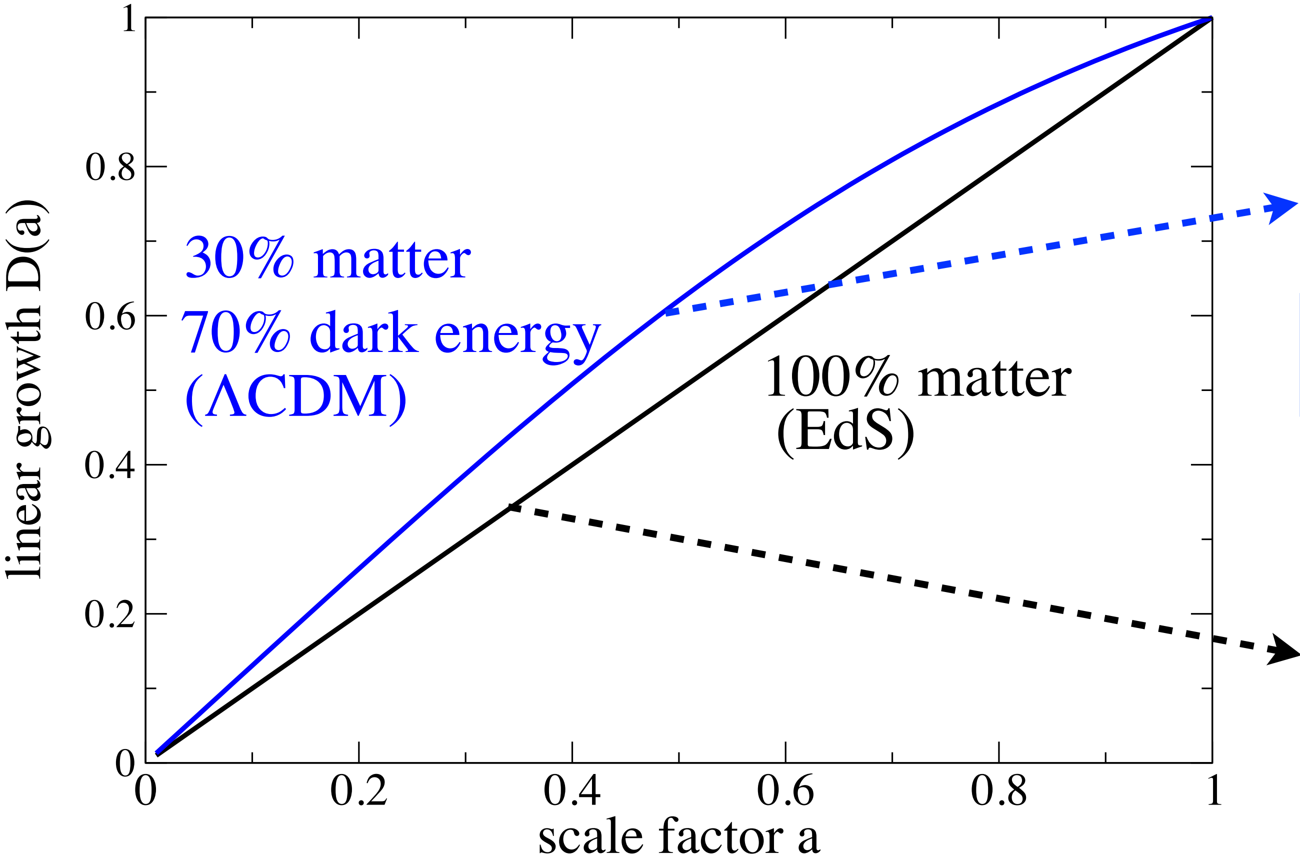
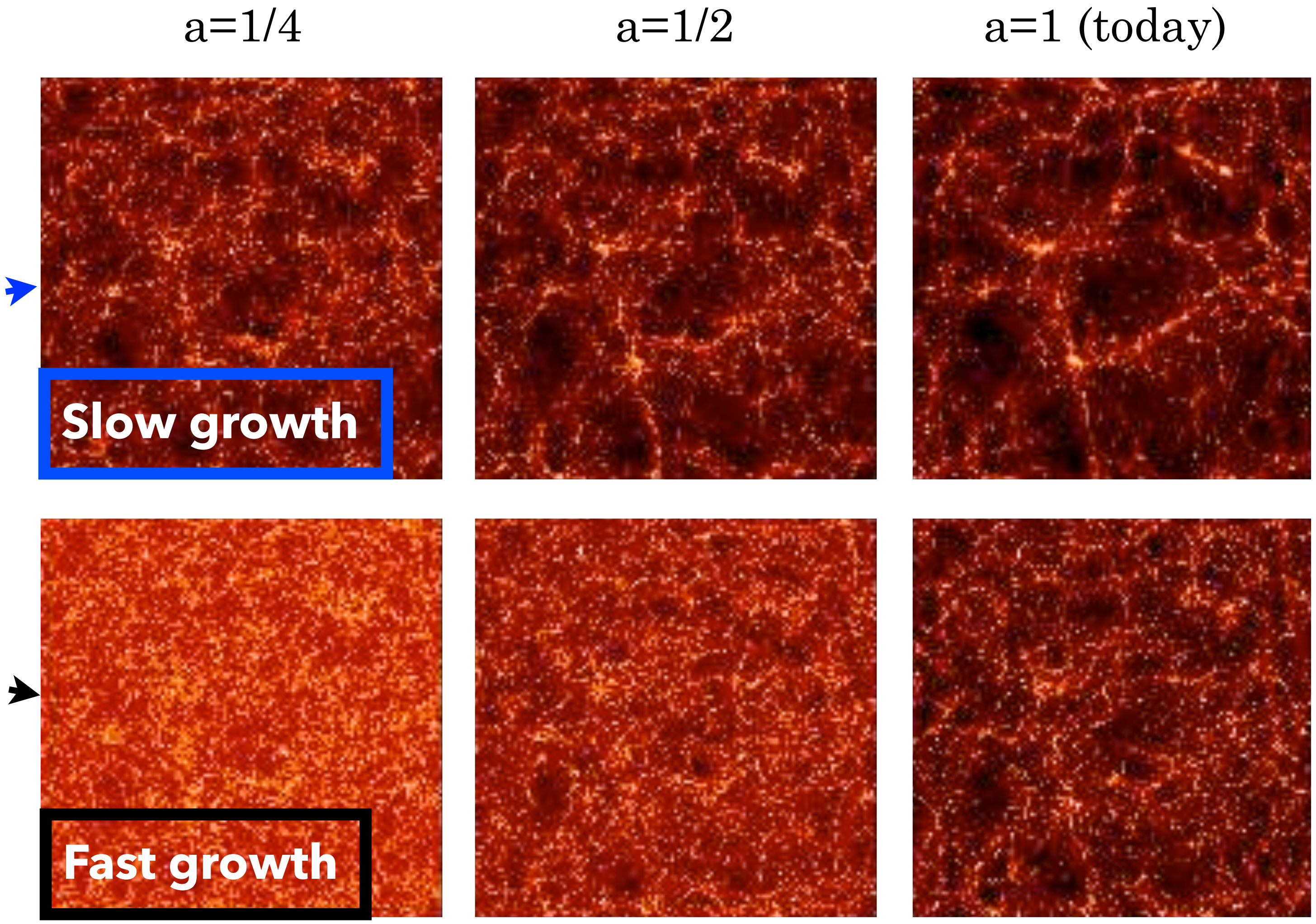


# Cluster Cosmology



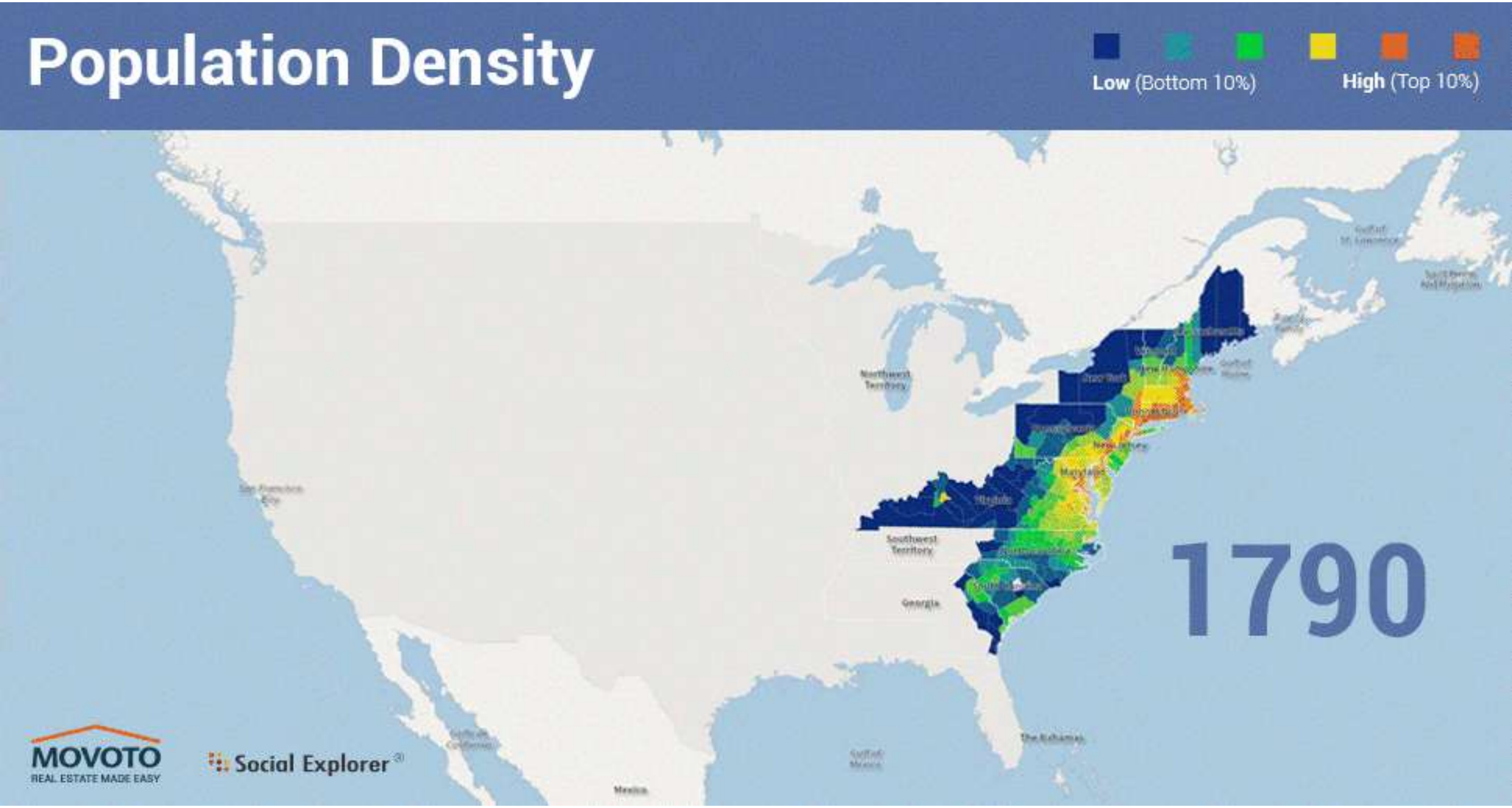




Huterer+14

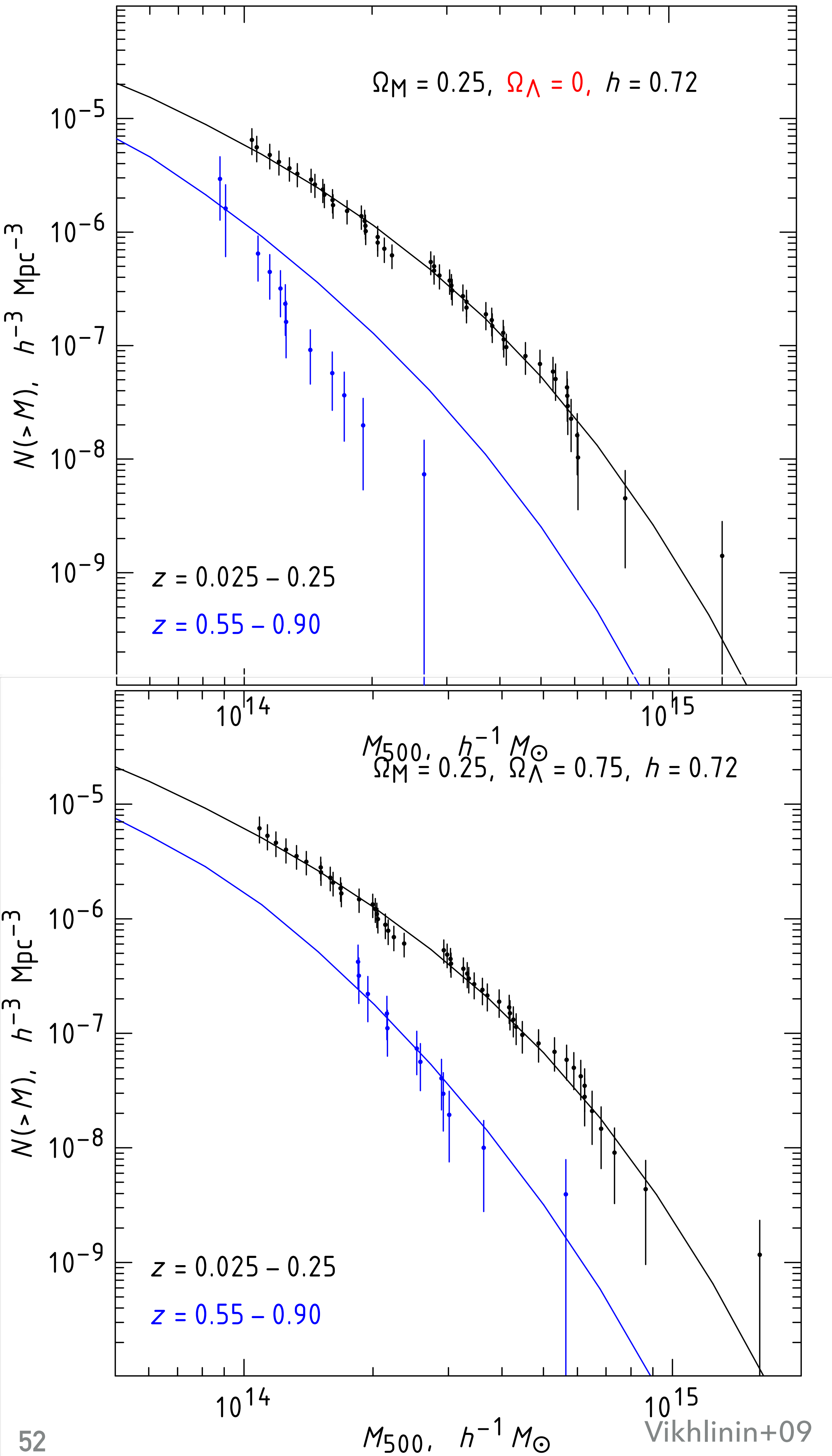
- Structure formations are extremely sensitive to dark energy.
- The number of galaxy clusters in a cosmic volume is powerful in constraining dark energy.

Counting clusters (abundance) to infer cosmology!

# Analogy of Cluster Cosmology



# Halo Mass Function

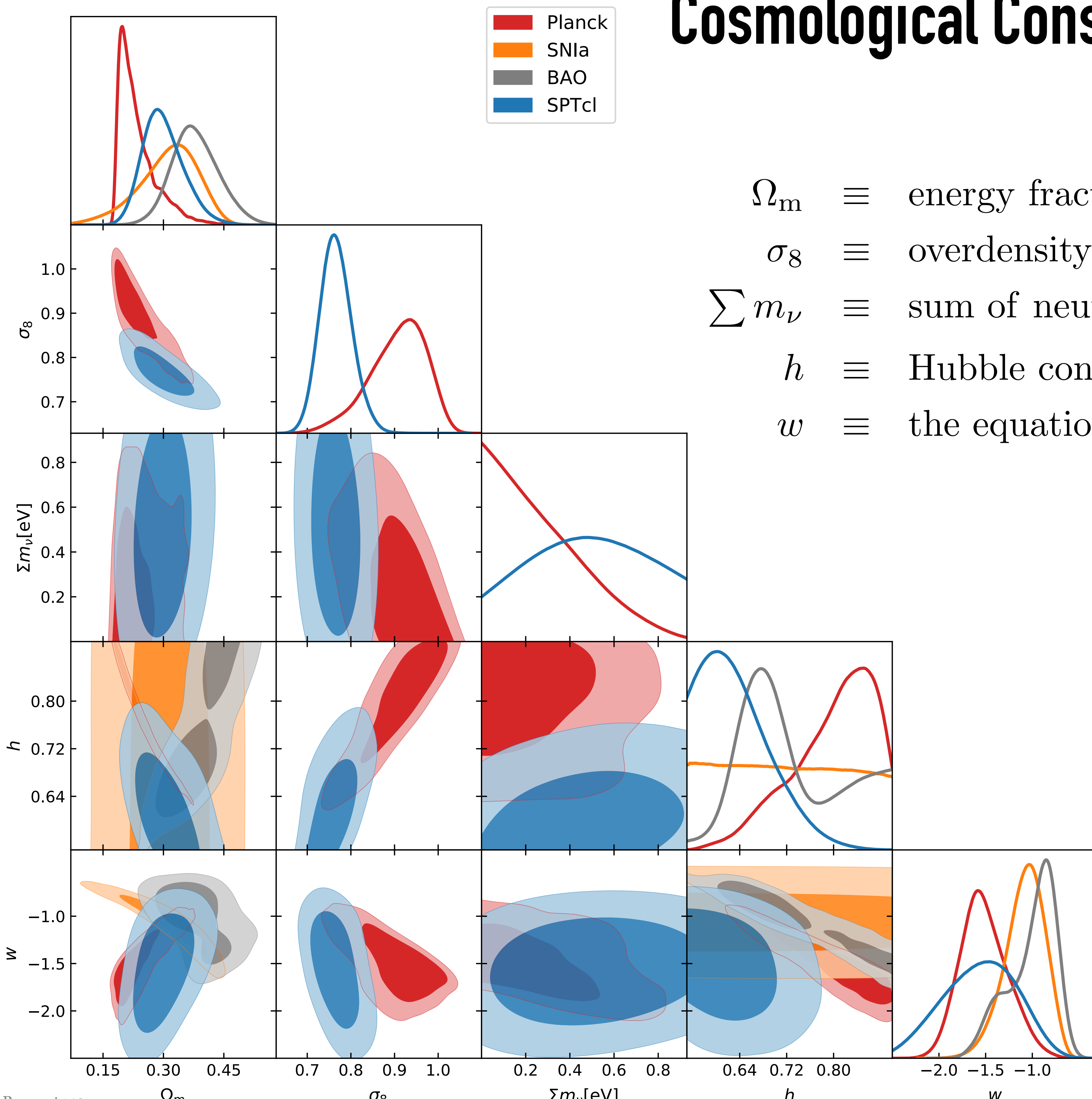



Vikhlinin+09

$$\frac{\mathrm{d}n}{\mathrm{d}M} = f(\sigma)\,\frac{\rho_{\mathrm{m}}}{M}\,\frac{\mathrm{d}\sigma^{-1}}{\mathrm{d}M}$$

$$\sigma \equiv \frac{1}{(2\pi)^3}\int P(k,z)W_M(k)^2\mathrm{d}k^3$$

# Cosmological Constraints From Clusters




Bocquet+19

$$\Omega_\mathrm{m} \equiv \text{energy fraction of matter}$$

$$\sigma_8 \equiv \text{overdensity fluctuations at 8 Mpc}/h$$

$$\sum m_\nu \equiv \text{sum of neutrino masses}$$

$$h \equiv \text{Hubble constant, i.e., } H_0 = h \times 100\frac{\text{km/sec}}{\text{Mpc}}$$

$$w \equiv \text{the equation of state of dark energy}$$

# Take-Home Messages

- The standard cosmological model is introduced:
  - The universe originated from a big band ≈ 14 Gyr ago and has been expanding since then.
  - The cosmic expansion is accelerating at the present day.
  - The universe is well described by the ΛCDM model:
    - ≈5% baryonic matter
    - ≈25% cold dark matter (CDM)
    - ≈70% dark energy (Λ)
- The universe is homogeneous and isotropic at large scales, well described by the Friedmann equations.
- Cosmic structures of the universe act as linear perturbations to the uniform background.
- We have showcased some observational constraints on cosmology.

# Further Reading

- A very nicely written textbook at an undergraduate level:
  "**Introduction to Cosmology**" by Barbara Ryden
  https://www.amazon.com/Introduction-Cosmology-Barbara-Ryden/dp/1107154839
  Online lectures by Prof. Ryden:
  https://www.youtube.com/watch?v=ndSD9U34-gM&list=PLwWRX55-E1nYlD7o6W91wV8OYHongFNxU
- For those who want to dig more:
  "**Modern Cosmology**" by Scott Dodelson is a must-have:
  https://www.amazon.com/Modern-Cosmology-Scott-Dodelson/dp/0128159480