2023 NCTS-TCA summer student program workshop

Astrophysical fluid dynamics
- a brief introduction -

Hung-Yi Pu (National Taiwan Normal University) July 3th 2023

what is a fluid?

Astrophysical fluid dynamics?

Astrophysical fluid dynamics

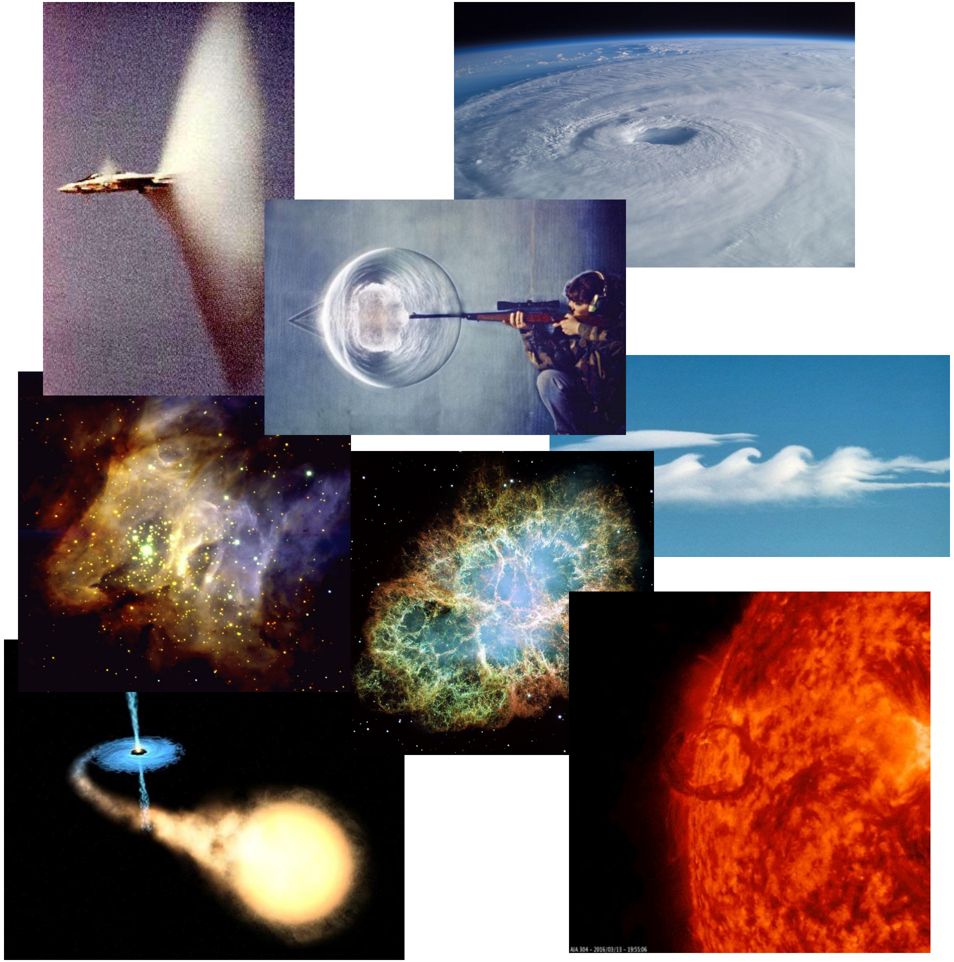
- most of the baryonic matter (consists of three quarks) in the Universe can be treated as a fluid
- the liquid phase less common
- plasma: magnitized fluid, interacting via EM interactions
- gravity is important
- large scale

reference

- Astrophysical Flows by Pringle & King
- Principle of astrophysical fluid dynamics by Clarke & Carswell
- The physics of **plasmas** by Boyd & Sanderson
- The physics of fluids and **plasmas** by Choudhuri
- The physics of astrophysics volume II: **gas dynamics** by Shu
- Fluid Mechanics by Frank M. White

outline

- hydrodynamics
  - shear and viscosity
  - velocity field
  - governing equations
    - continuity
    - momentum
    - energy
    - (equation of state)
  - turbulence and energy cascade
  - shock
- magnetohydrodynamics (MHD)
  - plasma
  - ideal MHD
  - astrophysical applications

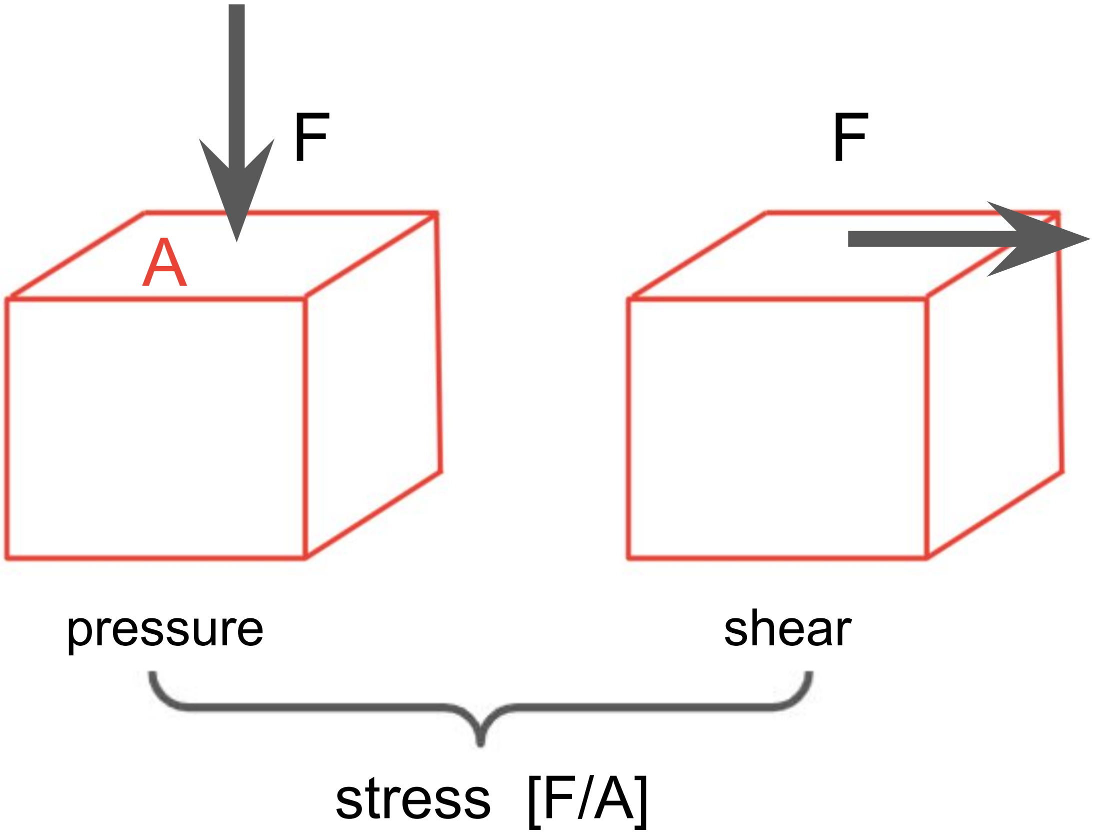


what is a fluid?
When can we apply the concept of fluid?

stress and shear

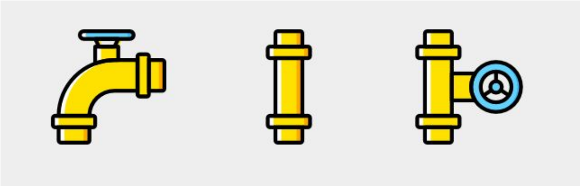
typical definition of fluid:

can move under the action of a **shear stress**, no matter how samll that stress may be



classification of fluid

- materials
  - solid
  - fluid
    - liquid
    - gas
      - statics (fluid statics/ hydrostatics)
      - in motion
        - internal flows

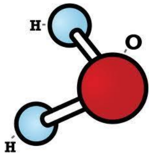


        - external flows

Newtonian	vs.	non-Newtonian
incompressible		compressible
inviscid		viscous
laminar		trubulent
steady		unsteady
one phase		multi phase

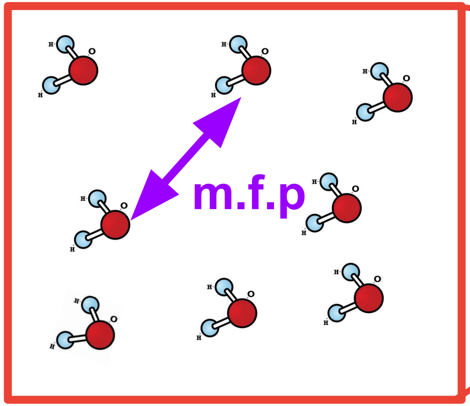
fluid: a macroscopic approach

different levels: particle → distribution funciton→ continuum model (**L** >> **m.f.p**)

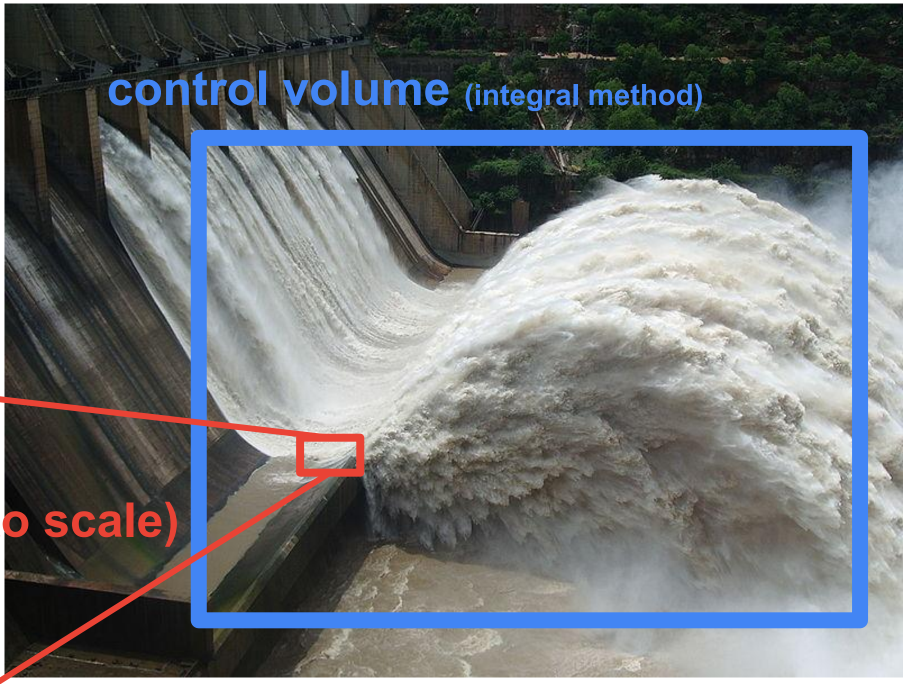


L

fluid element (differential method)



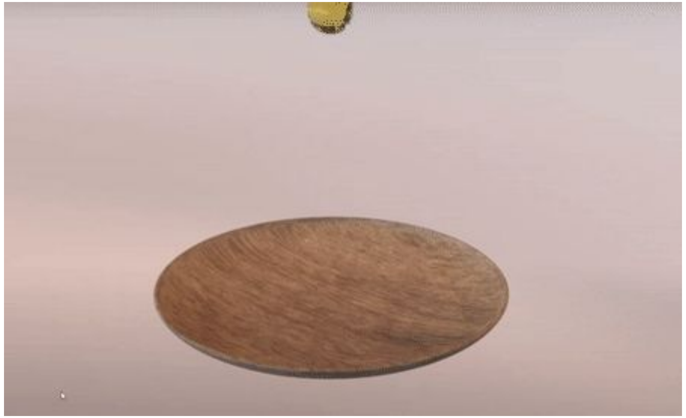
(not to scale)



mean free path (m.f.p): average distance a particle travels before it colldes with another particle

from water molecular to river

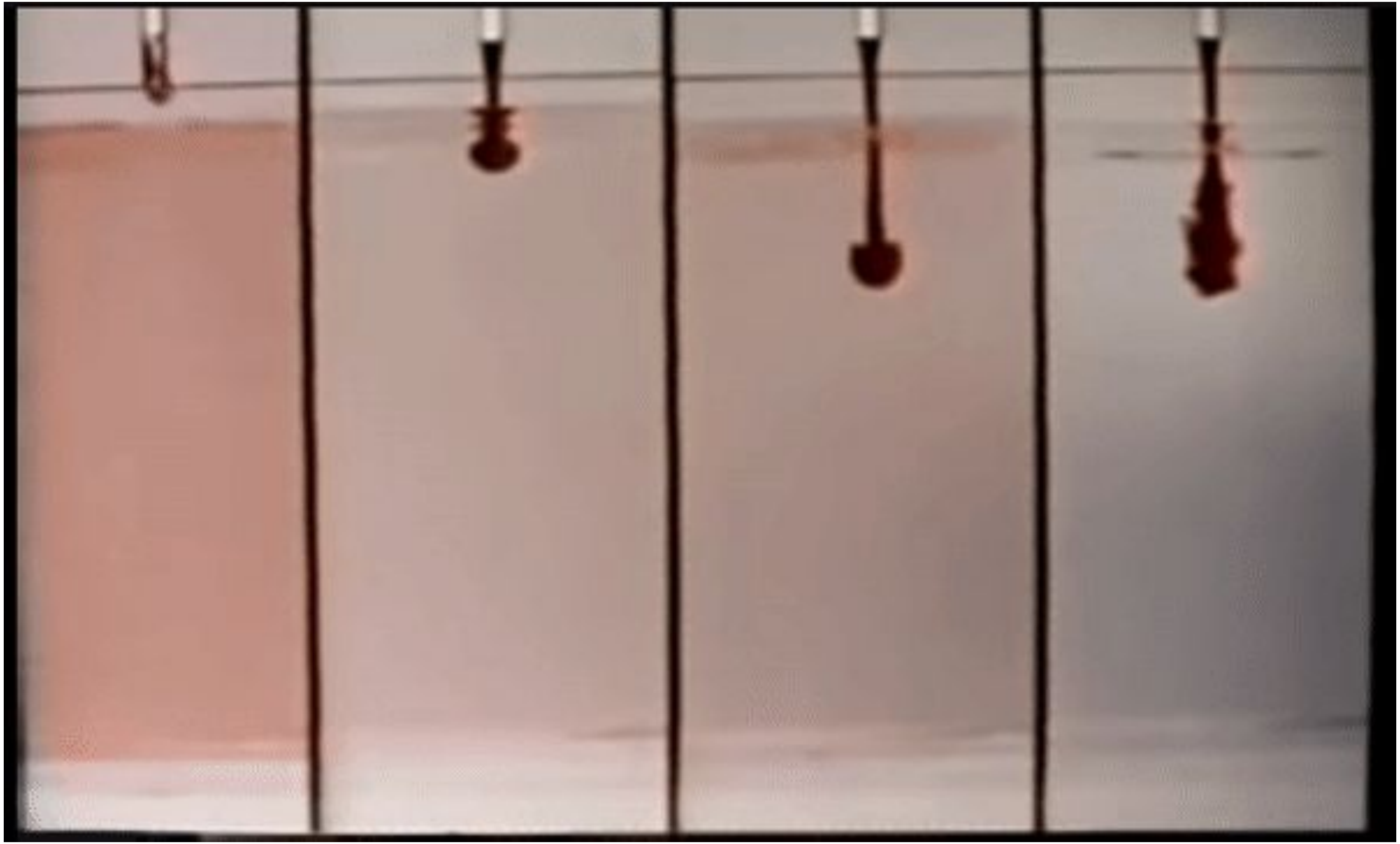
why fluid dynamics is hard?



simluation or real honey?
credit: https://www.youtube.com/watch?v=I3c4m29coB4

simluation
credit: wiki

non-linear + using one equation (Navier-Stokes equation) to descibe all personality (= viscosity) of different fluids!



Millennium Prize Problems (千禧年大獎難題)

seven unsolved problems in mathematics that were stated by the Clay Mathematics Institute on May 24, 2000.

A correct solution to any of the problems results in a

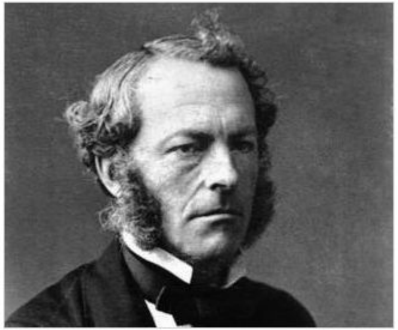
US$1 million prize being awarded by the institute to the discoverer(s).

CMI

ABOUT PROGRAMS PEOPLE MILLENNIUM PROBLEMS PUBLICATIONS EVENTS NEWS

does the solution existence? is it smooth?

Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). Public Domain

This problem is: Unsolved

Rules:

Rules for the Millennium Prizes

Related Documents:

Official Problem Description

Related Links:

Lecture by Luis Cafarelli

*hypothetical water with no viscosity was named “dry water” by Richard Feynman.

“when I meet God, I am going to ask him two questions: Why relativity? and why trubulence?

I really belive he will have an answer for the first.”

W. Heisenberg (1907-1976)

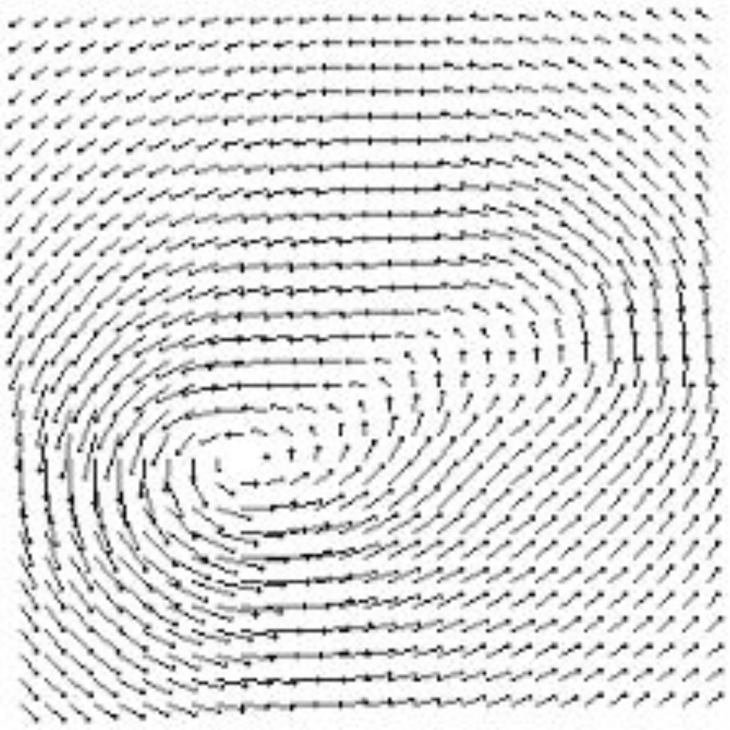
notation

- for 3D flow in cartesian coordinate

$$(V_x, V_y, V_z) = (u, v, w)$$

- for 2D flow

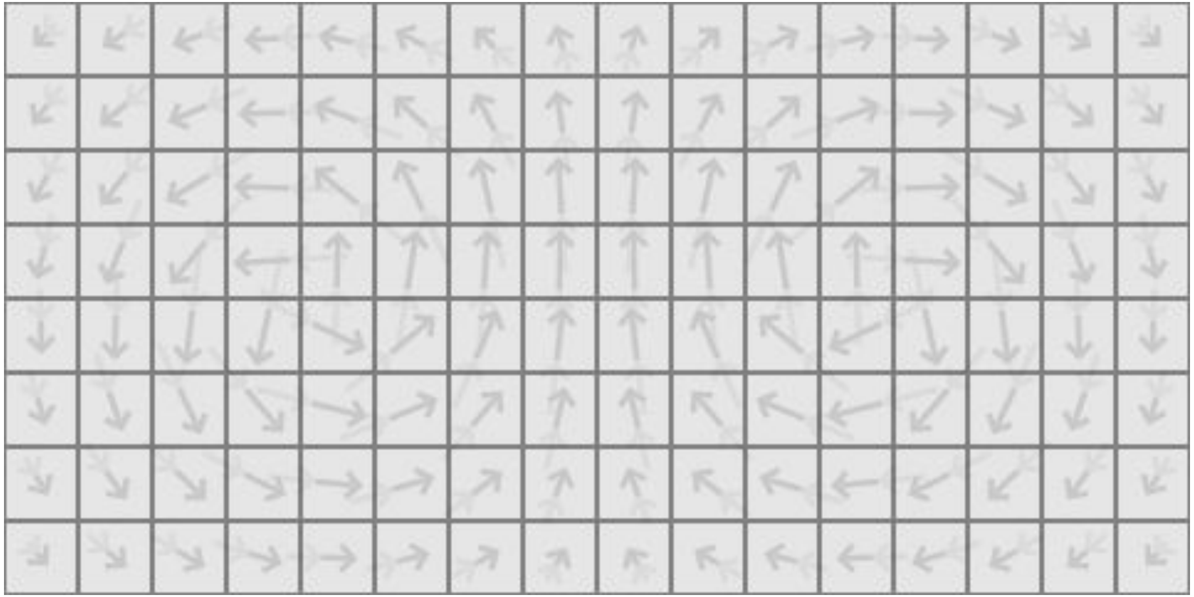
$$(V_x, V_y) = (u, v)$$



fluid dynamics as velocity field + fluid properties

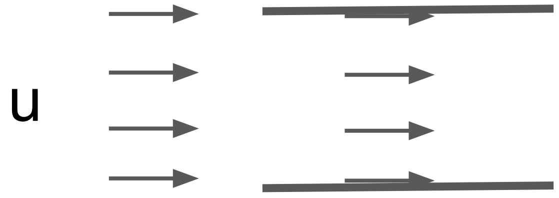
t1: velocity field

t2: velocity field



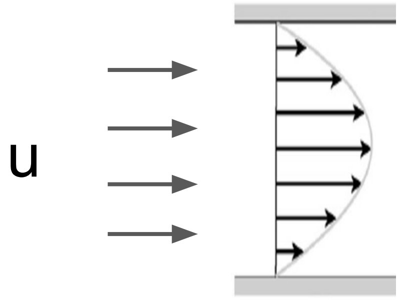
viscous and invicid (steady) flow

invicid flow



does not exist!

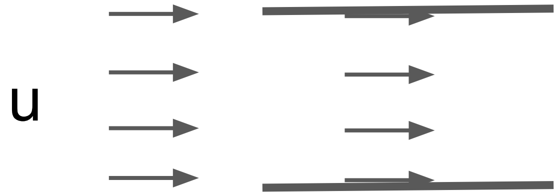
viscous flow



Poiseuille flow

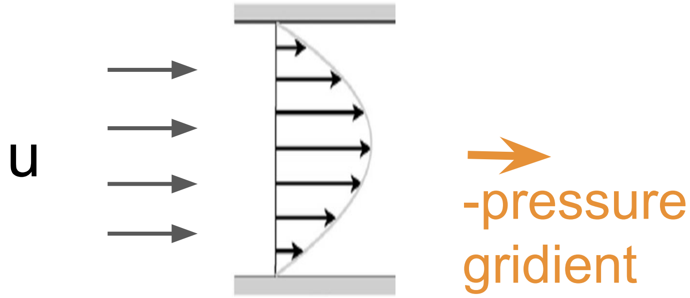
viscous and invicid (steady) flow

invicid flow

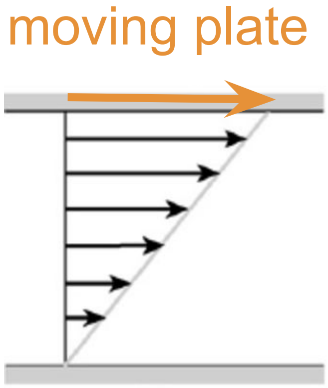


does not exist!

viscous flow



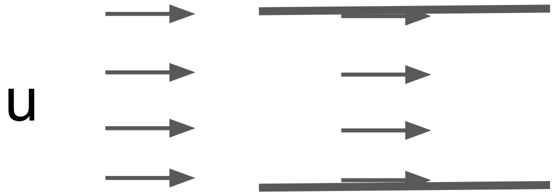
Poiseuille flow



Couette flow

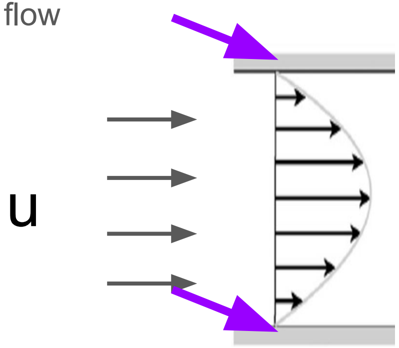
viscous and invicid (steady) flow

invicid flow

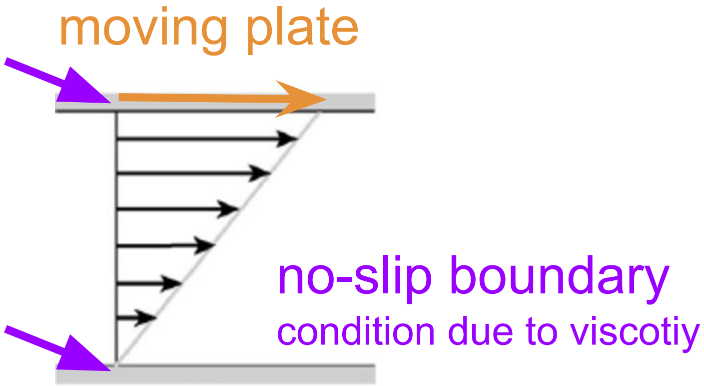


does not exist!

viscous flow

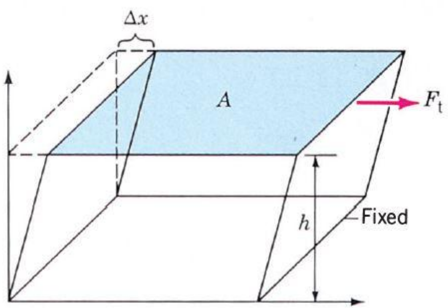


Poiseuille flow

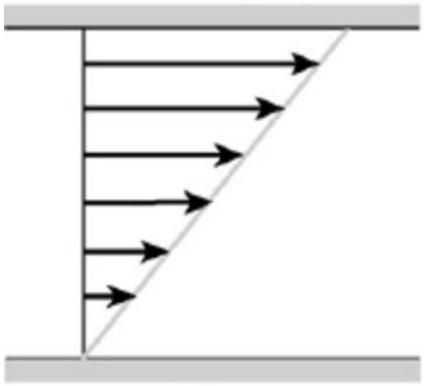


Couette flow

Shear Modulus



$$S = \frac{F_t / A}{\Delta x / h}$$



$$\text{(dynamical)viscosity} = \frac{F_t/A}{\Delta v_x/h}$$

shear

(shear) stress causes strain (via viscosity)

meassure of the resistence of a fluid to gradual deformations by shear stress

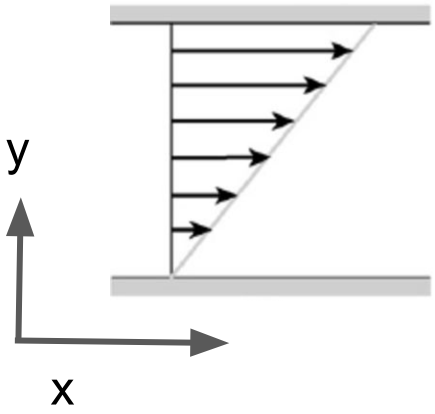
dynamic viscosity

$$\tau = \mu \frac{\partial u}{\partial y}$$

shear stress [F/A]

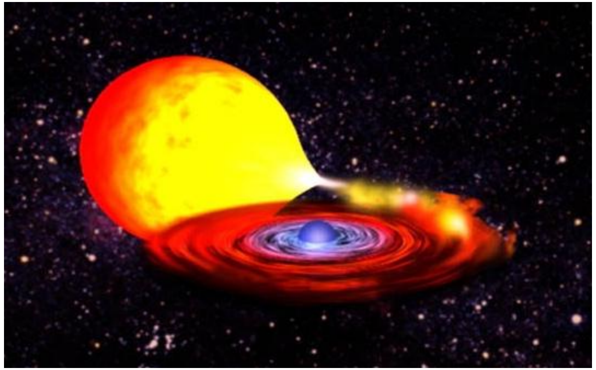
shear/strain rate [1/s]

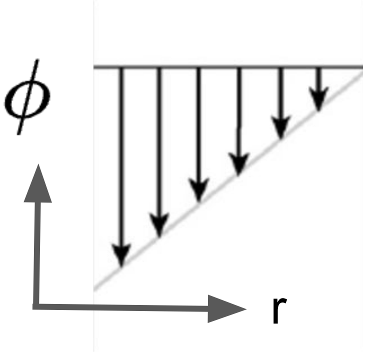
stress = viscosity x strain



$$\nu = \frac{\mu}{\rho}$$

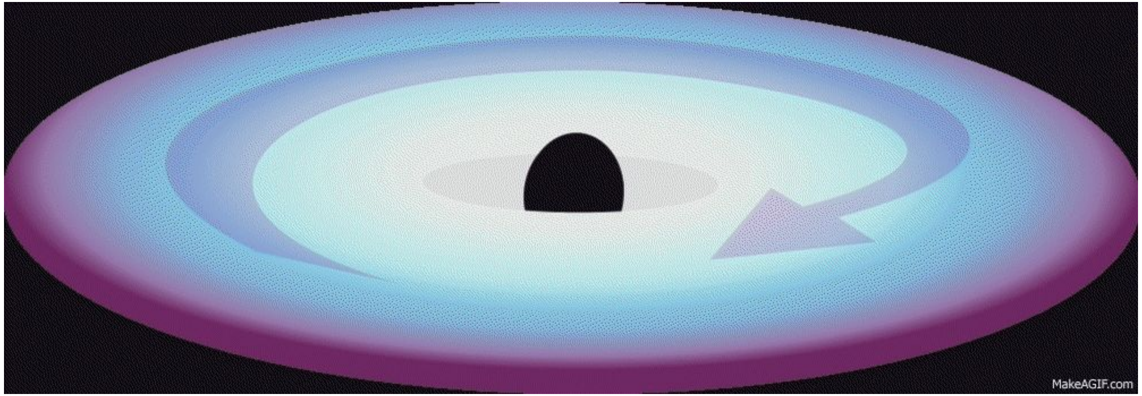
kinetic viscosity [VL]





$$\omega_{\text{Kepler}} = \left(\frac{GM}{r^3}\right)^{1/2}$$

$$\alpha - \text{disk}:$$
$$\nu = \alpha H C_s$$



identify the shear:

(a) $$\nu\rho\left(\frac{\partial r\omega}{\partial r}\right)$$

(b) $$\nu\rho\left(r\frac{\partial \omega}{\partial r}\right)$$

(a) or (b) ? why?

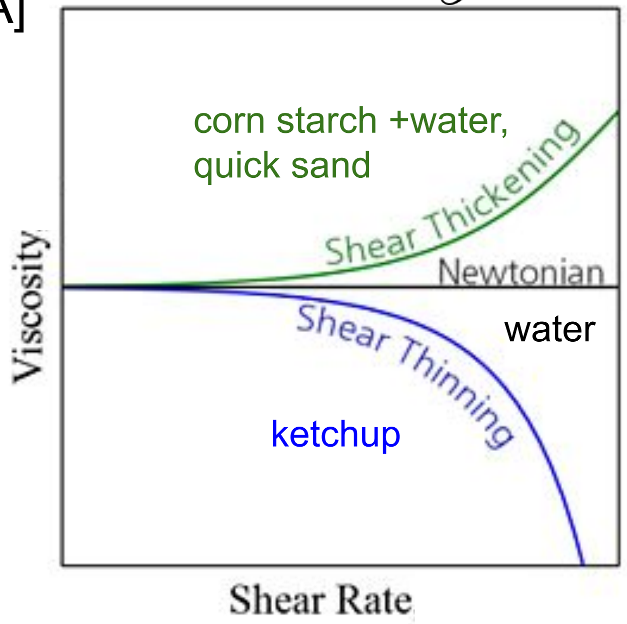
*note: shear should dissapear if $\omega = \text{constant}$

dynamic viscosity

$$\tau = \mu \frac{\partial u}{\partial y}$$

shear/strain rate [1/s]

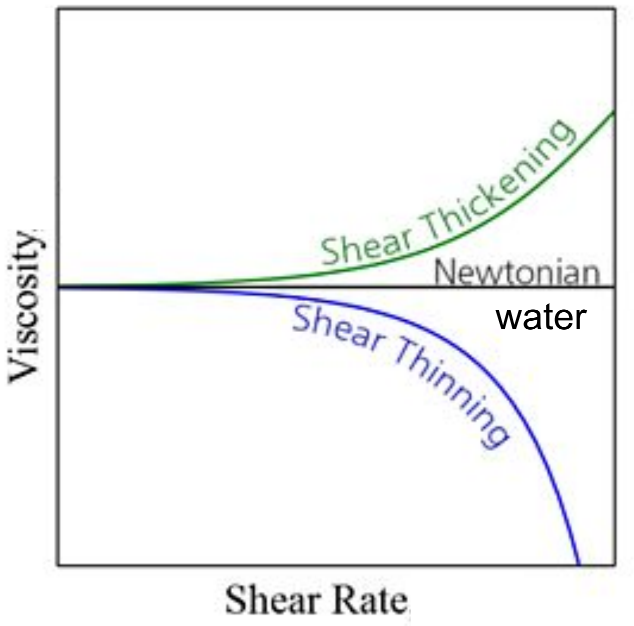
shear stress [F/A]



movie credit: 國立台中教育大學 NTCU
科學教育與應用學系

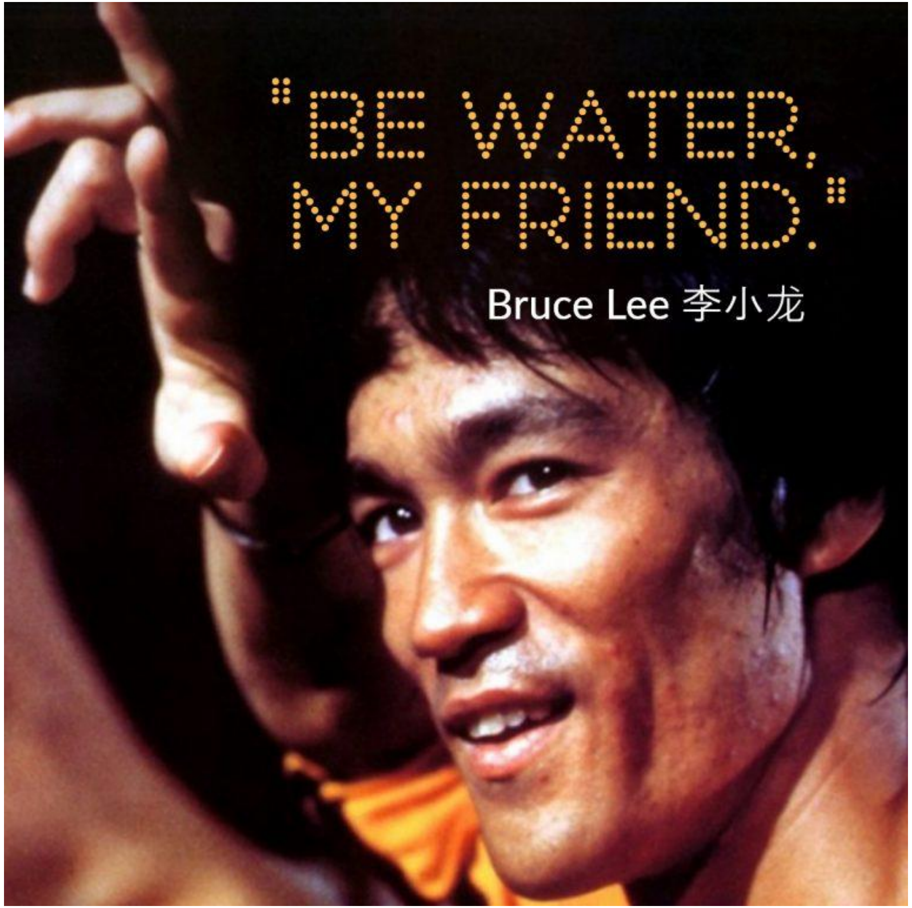
shearing thickening!

Ketchup: shearing thinning!

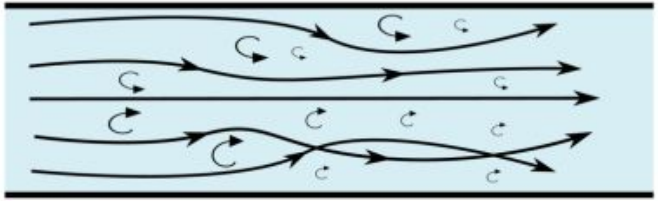


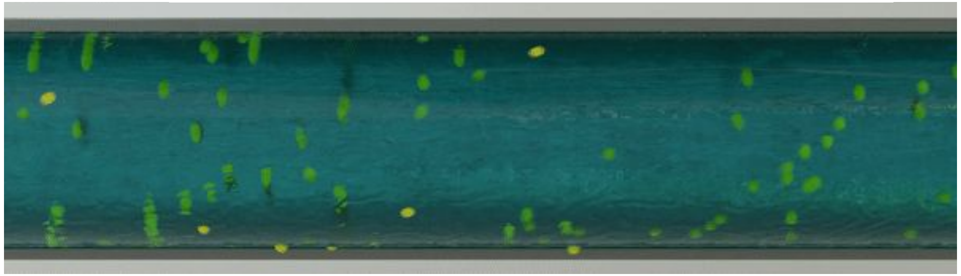
不卑不亢

Neither humble nor overbearing



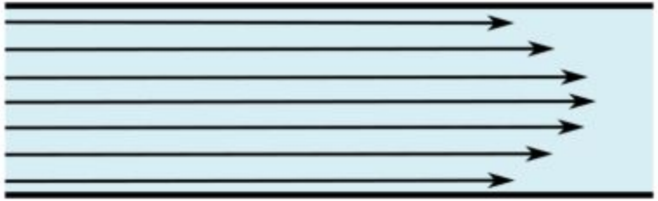
(unsteady) turbulent flow

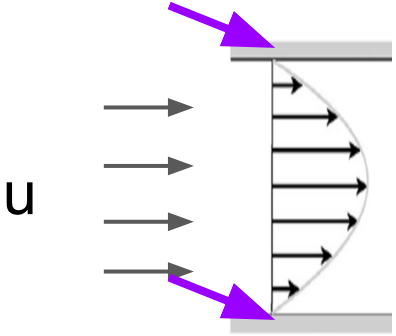




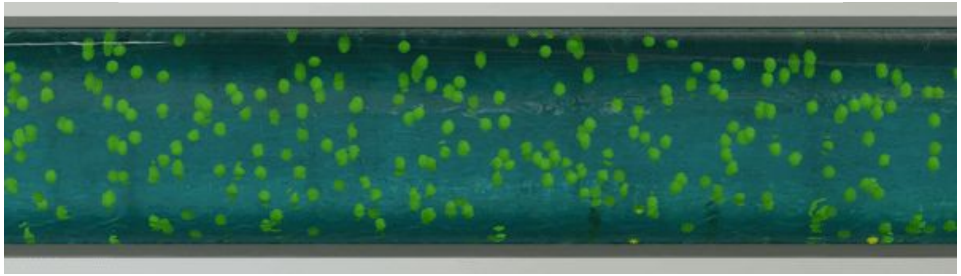
keyword: mixing!

laminor flow



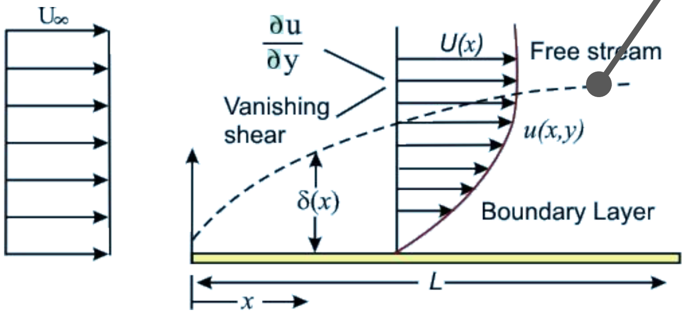


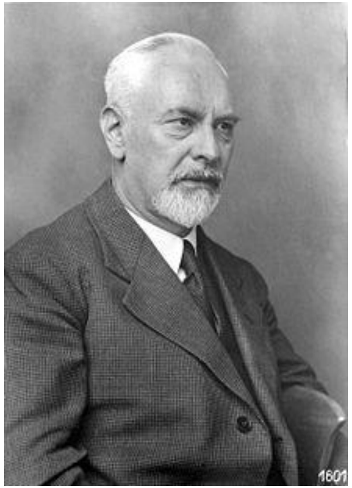
Poiseuille flow



external flow

$$U(x, y = \delta) = 0.99 U_\infty$$





Ludwig Prandtl

"**Prandtl's boundary layer theory**" solves the tension bt. experiment and theory: **outside the boundary layer, the viscous effect is not important!**

father of modern fluid mechanics!

invisid flow

↓

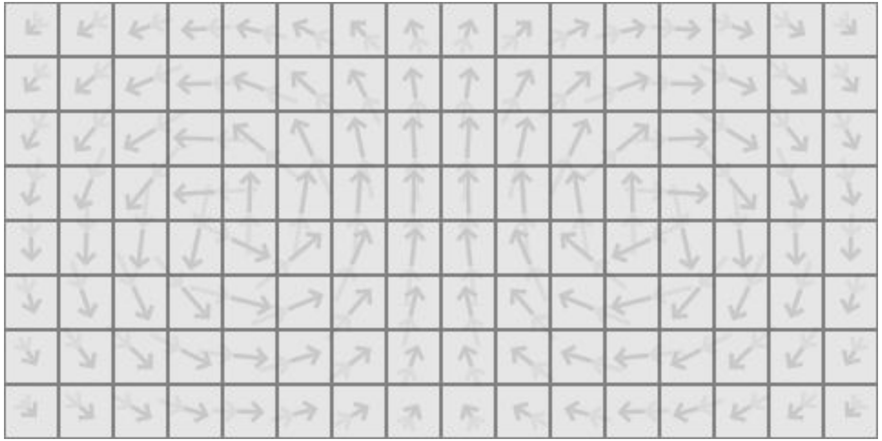
update velocity profile & boundary layer

↓

boundary layer displacement thickness momentum thickness etc.

*in numerical simulation (with good enough grids), everything is considered automatically

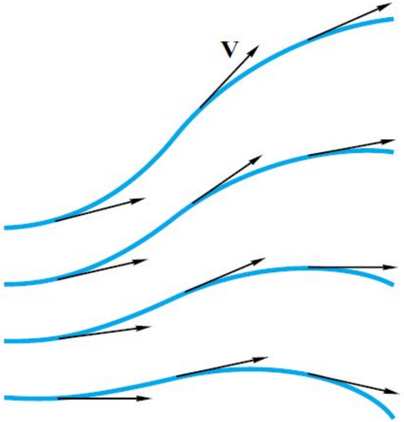
velocity field: V(u,v,w,t)

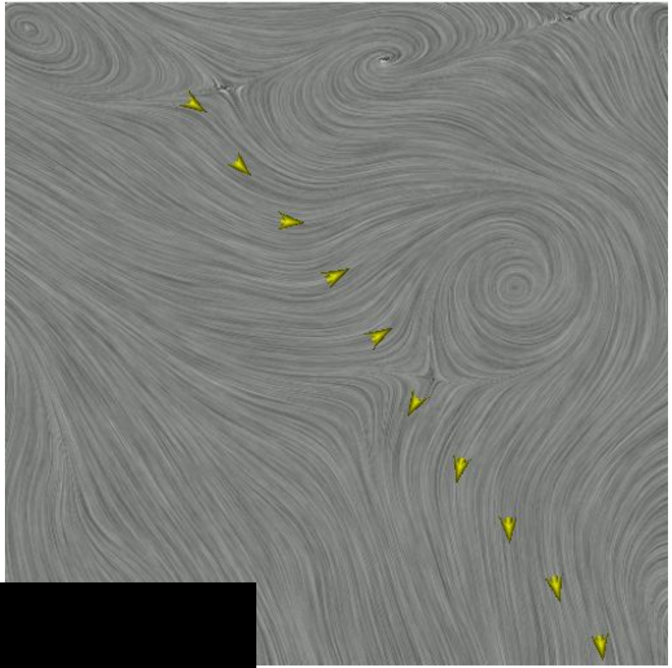


$\nabla \cdot \vec{V}$ relative change in volume per unit time — if =0: incompressible

$\nabla \times \vec{V}$ vorticity: measure of local rotation — if >0: counter-clockwise locally streamline (at constant t):

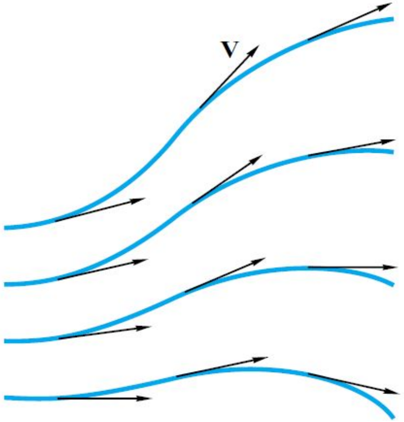
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}$$

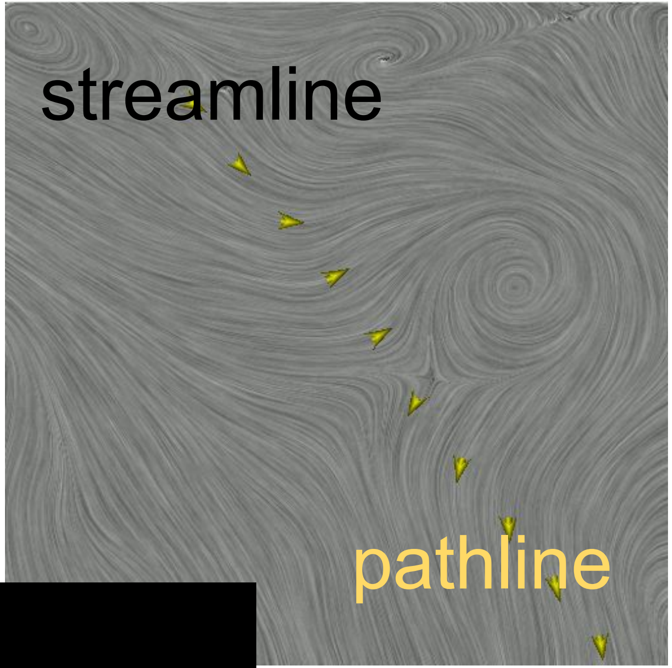




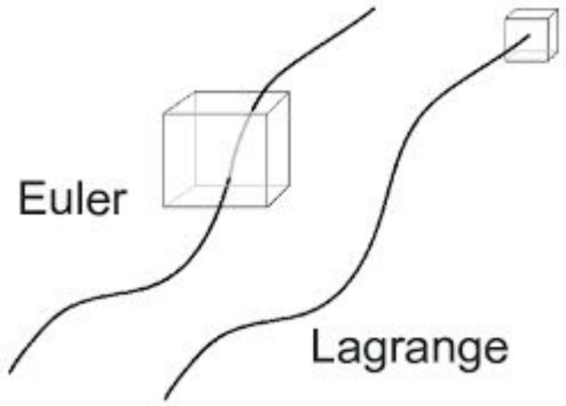
streamline (at constant t):

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}$$





a tale of two views (for everything!)



substantial/material derivative

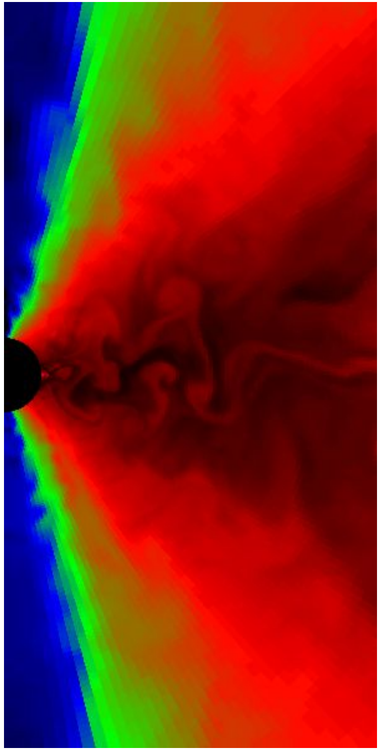
$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

LHS: Largragian point of view
(**ride on the particles**)

RHS: Eulerian point of view
(**stay at the fixed grid**)

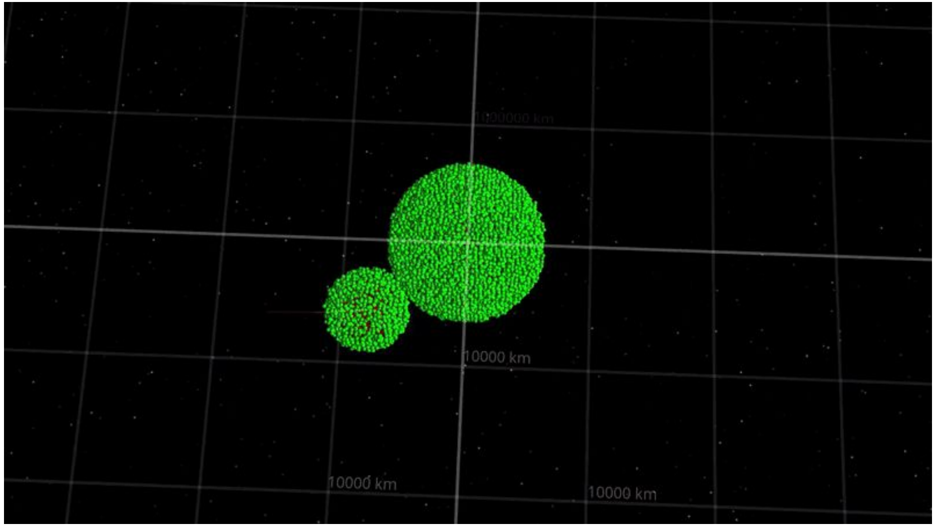

Eulerian
Lagrangian

grid-based simulation

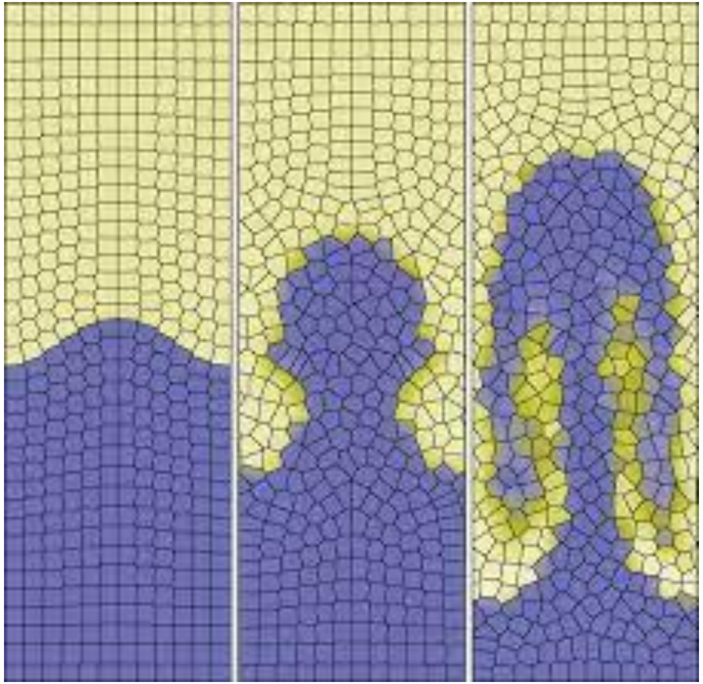


movie credit: Stone

particle-based simulation

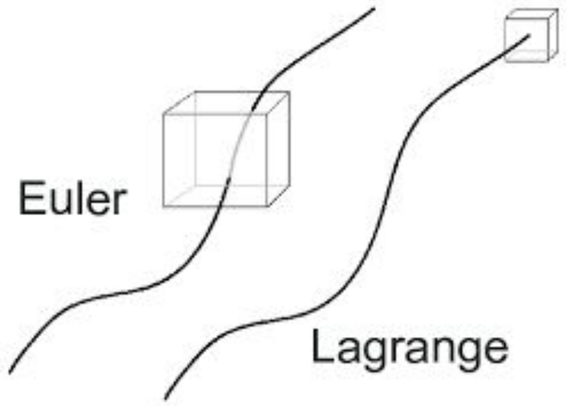


moving-mesh (Lagragian grid)



credit: Springel

a tale of two views



substantial/**material derivative**

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

<u>proof</u>

$$d\rho(t, x, y, z) = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz$$

$$\frac{d}{dt}\rho = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt})\rho$$

governing equation I:

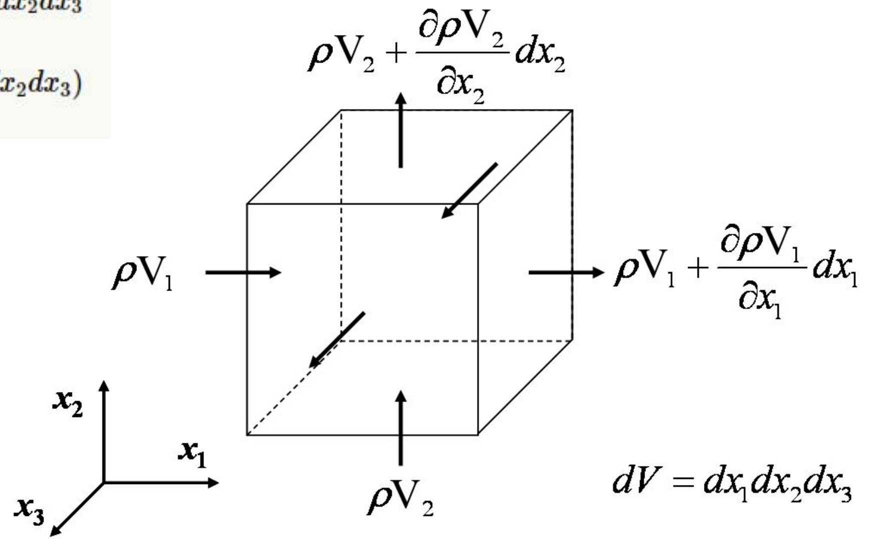
continuity equation (mass conservation)

conservation of mass (continity equation): differential form

$$\rho v_1(dx_2dx_3) + \rho v_2(dx_1dx_3) + \rho v_3(dx_1dx_2) - \left(\rho v_1 + \frac{\partial(\rho v_1)}{\partial x_1}dx_1\right)dx_2dx_3$$
$$-\left(\rho v_2 + \frac{\partial(\rho v_2)}{\partial x_2}dx_2\right)dx_1dx_3 - \left(\rho v_3 + \frac{\partial(\rho v_3)}{\partial x_3}dx_3\right)dx_1dx_2 = \frac{\partial}{\partial t}(\rho dx_1dx_2dx_3)$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$



conservation of mass (continity equation): differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\longleftrightarrow$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

$$(\mathbf{v} \cdot \nabla)\rho + \rho(\nabla \cdot \mathbf{v})$$

if incompressible $\left(\frac{D\rho}{Dt} = 0\right)$:

$$\boxed{\nabla \cdot \mathbf{v} = 0}$$

1D, steady, incompressible fluid is trivial!

conservation of mass (continity equation): integral form

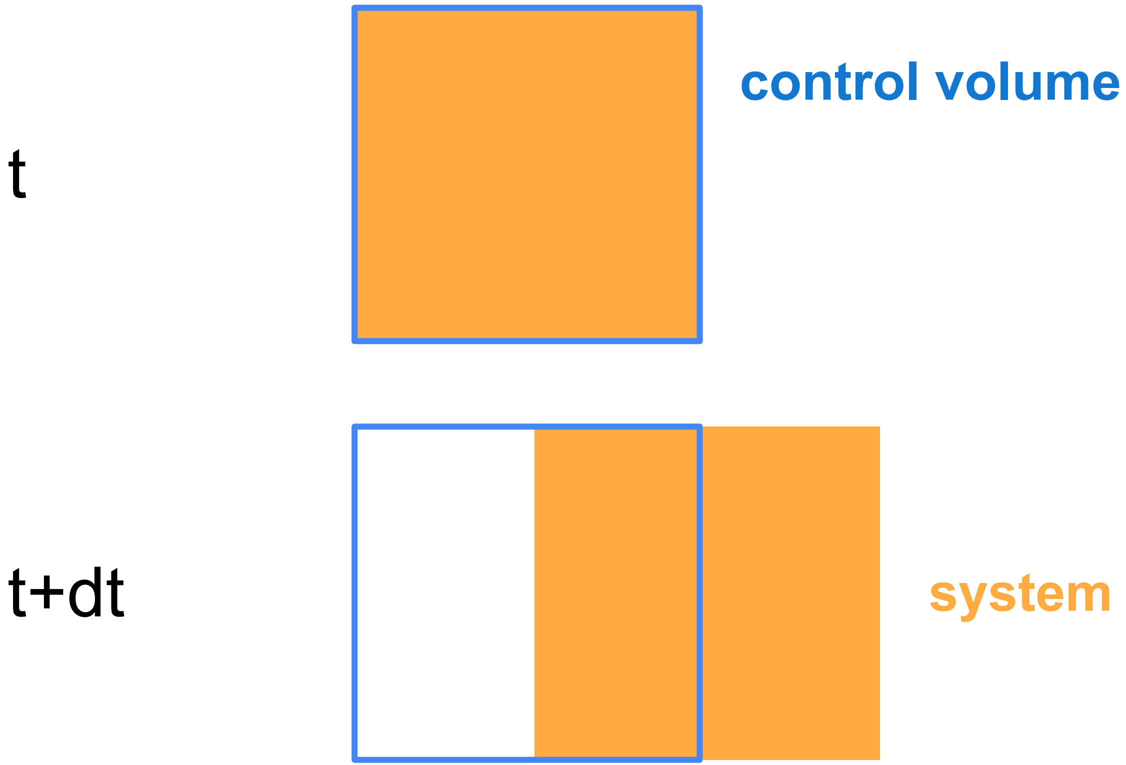
from Reynolds transport theorem: $\boxed{\rho u A = \text{constant}}$

↑

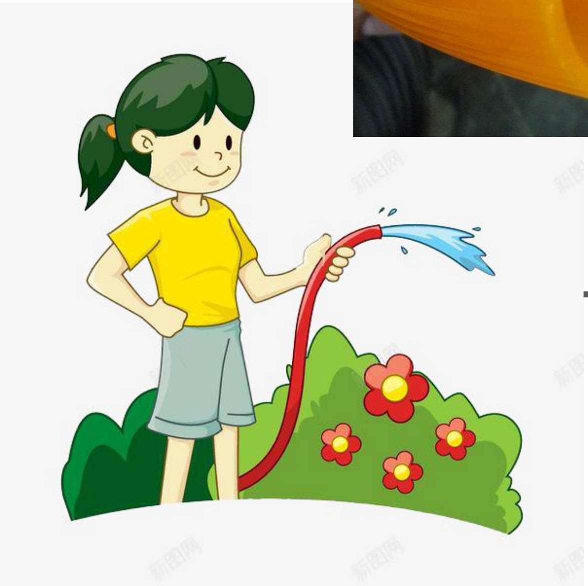
e.g. Fluid Mechanics by Frank M. White

if incompressible: $uA = \text{constant}$

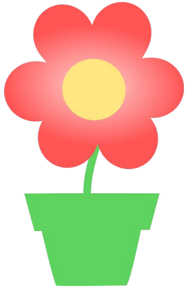
control volume and control surface



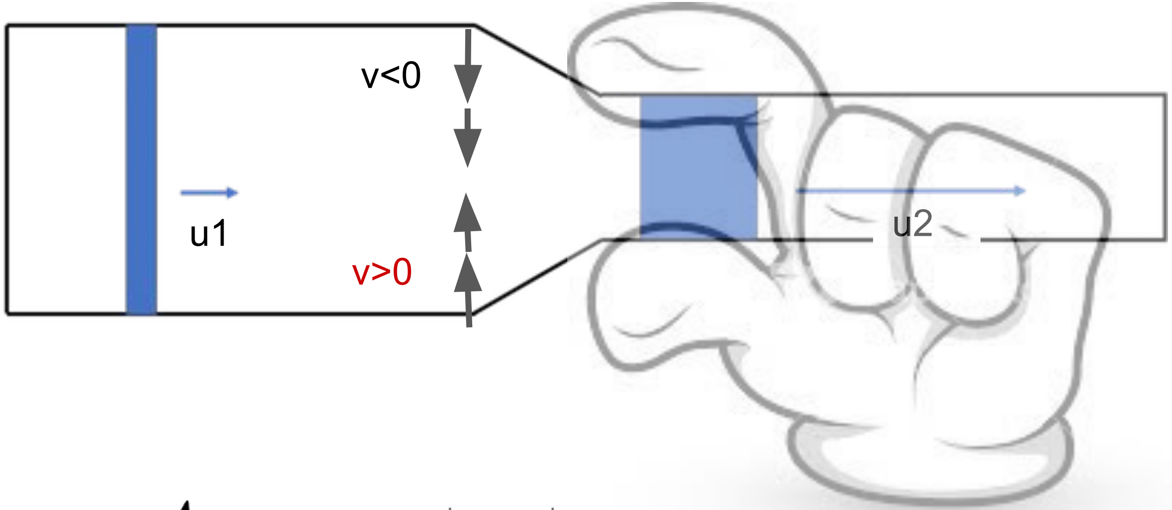
an example: 2D incompressible flow



what will you do?

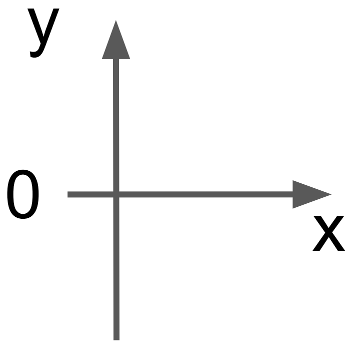


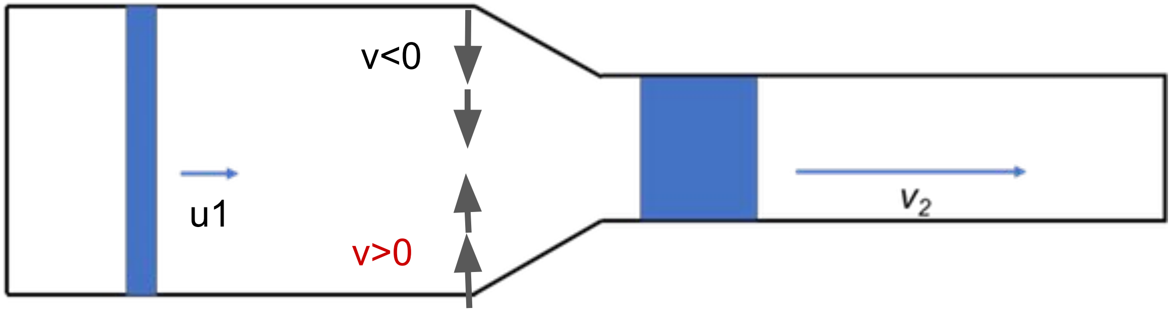
an example: 2D incompressible flow

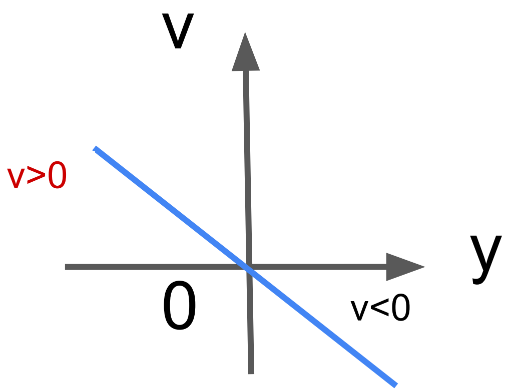


$$uA = \text{constant}$$

an example: 2D incompressible flow







$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

<0 (slope)

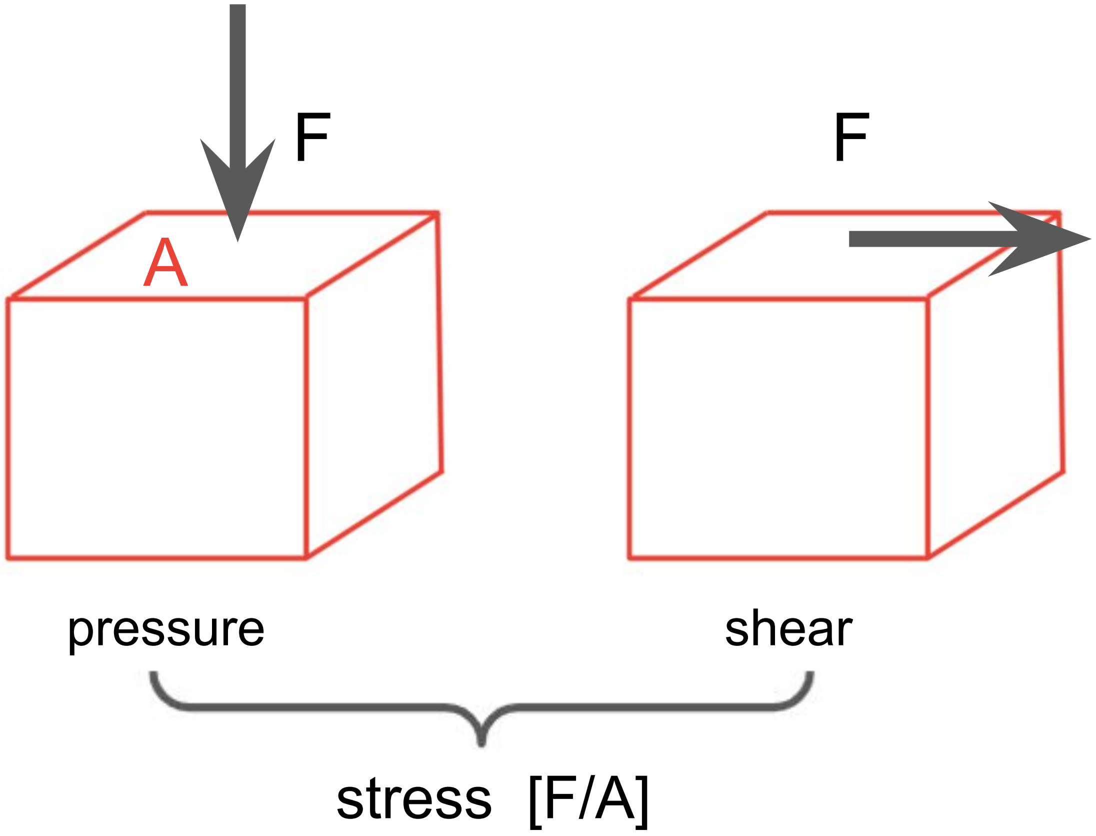
governing equation II:

momentum equation (Newton’s 2nd law)

stress and shear

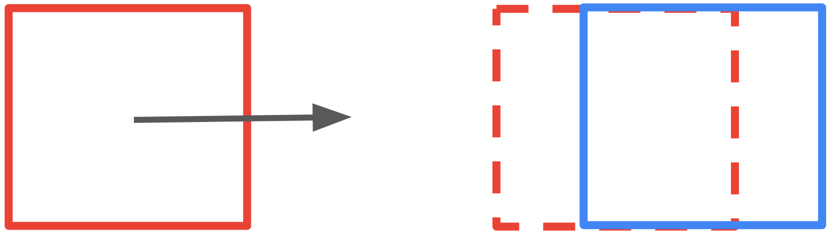
typical definition of fluid:

can move under the action of a **shear stress**, no matter how samll that stress may be



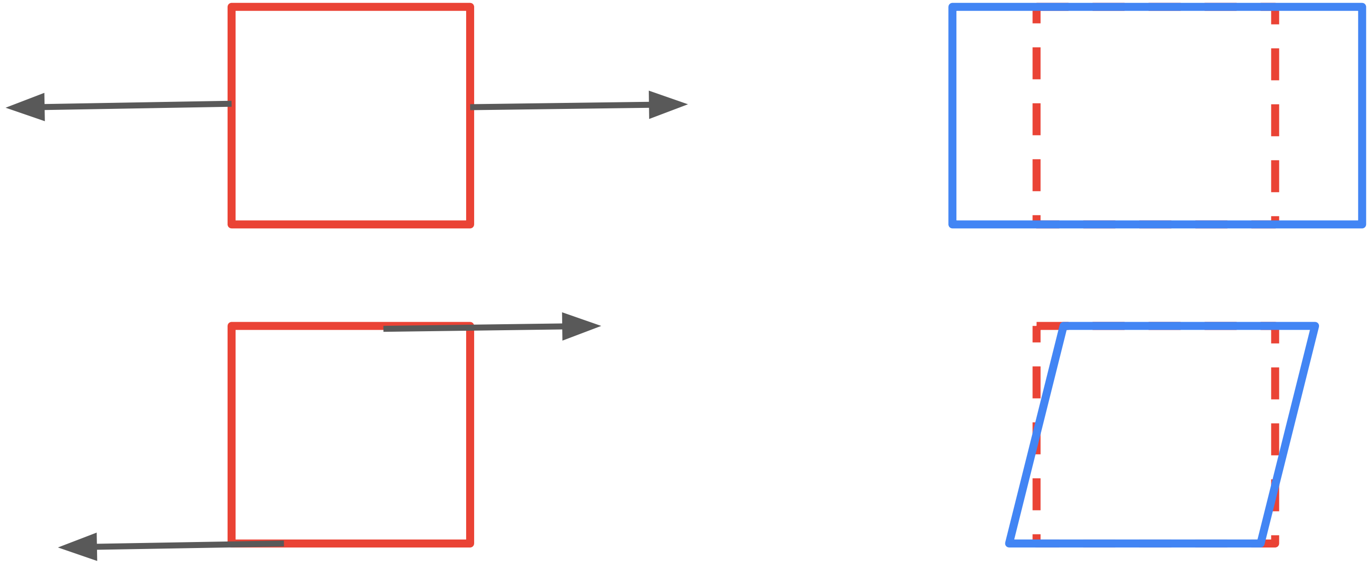
stress as a “surface force”

body force (acting on mass; does not require contact of the element)



surface force

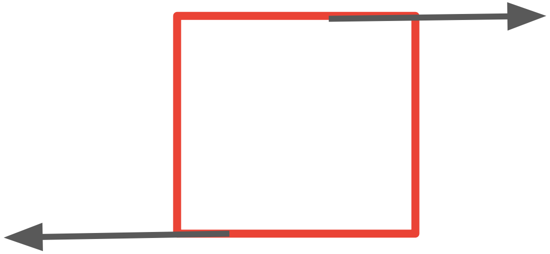
(acting on surface; requires contact of the element)

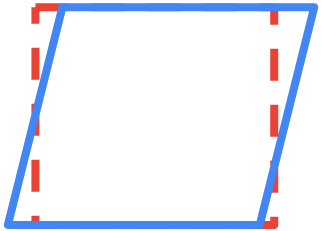


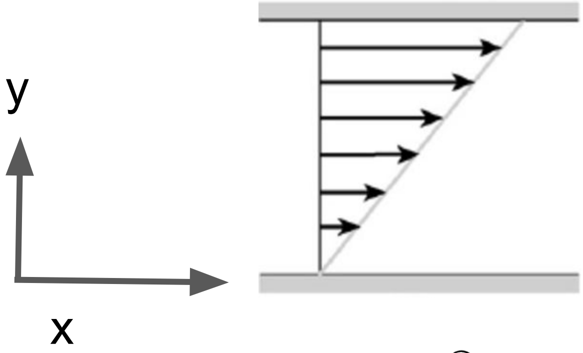
shear tensor

$$\tau_{ij}$$

surface — direction



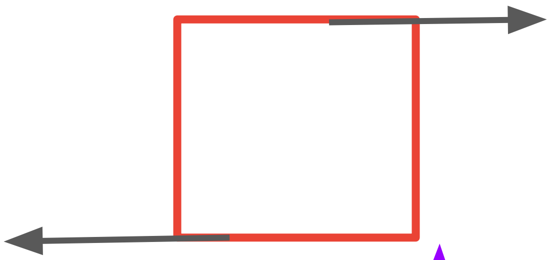


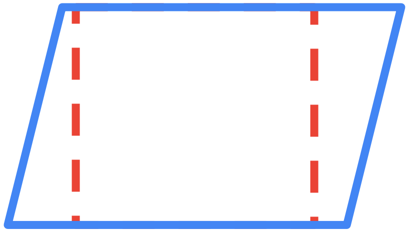


$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \quad ?$$

shear tensor τ_{ij}

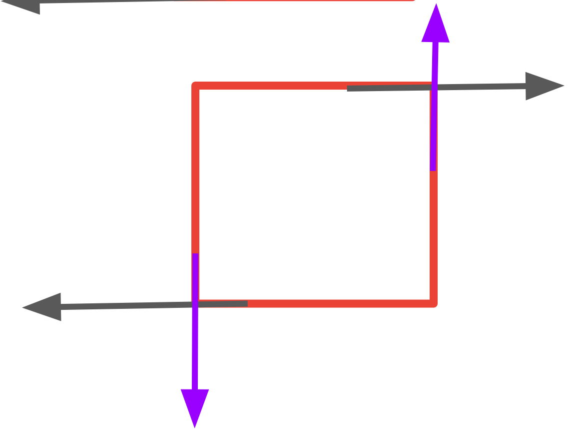
i: surface, j: direction

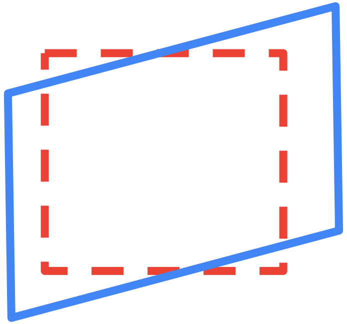




NO!

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \quad ?$$





$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

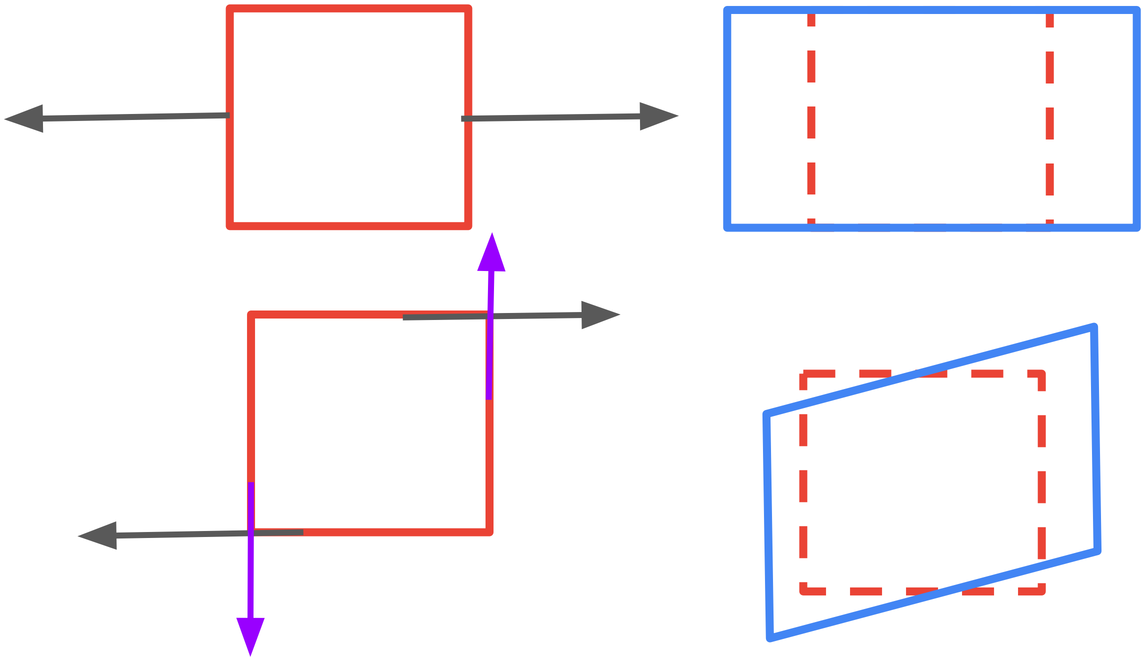
shear tensor

τ_{ij}

surface (i), direction (j)

net florce per unit volumn

$$\nabla \cdot \bar{\bar{\tau}} = \Sigma(\nabla_i \tau_{ij})$$



$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \vec{V})$$

$$\tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \tau_{xy}$$

ma = F

body force

gravity

EM

surface force

pressure

viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

normal

shear

Navier-Stokes equation

ma = F

- body force
  - gravity
  - EM
- surface force
  - pressure
  - viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

- normal
- shear

viscous term

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla P + \nabla \cdot \bar{\bar{\tau}}$$

$$= \rho \vec{g} - \nabla P + \mu \nabla^2 \vec{V}$$ if costant viscosity, imcompressible fluid

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

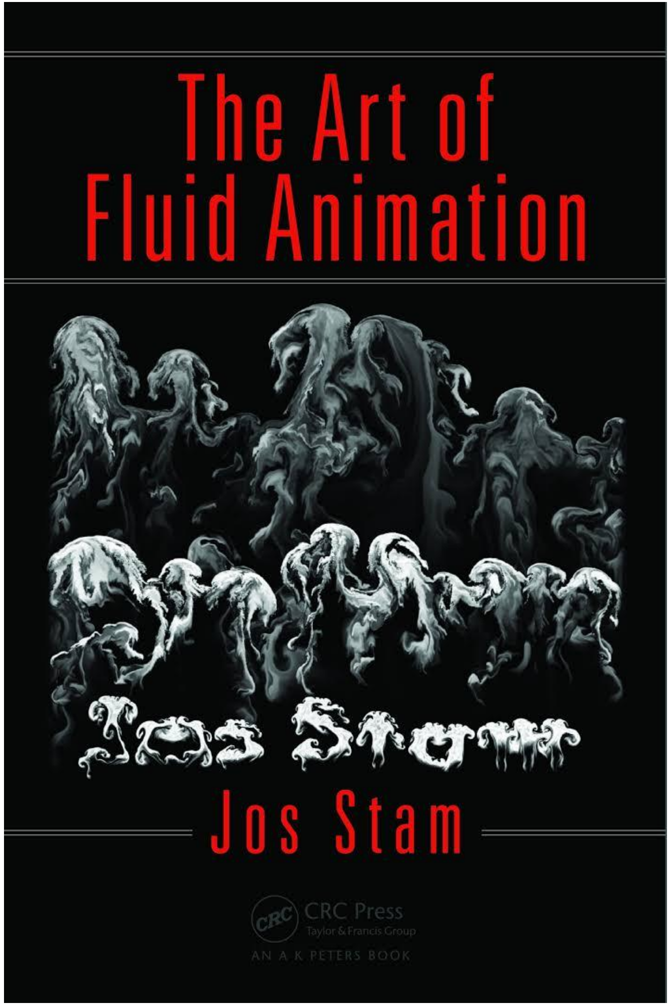
Real-Time Fluid Dynamics for Games

Jos Stam

Alias | wavefront
210 King Street East
Toronto, Ontario, Canada M5A 1J7
Email: jstam@aw.sgi.com,
Url: http://www.dgp.toronto.edu/people/stam/reality/index.html.

Abstract

In this paper we present a simple and rapid implementation of a fluid dynamics solver for game engines. Our tools can greatly enhance games by providing realistic fluid-like effects such as swirling smoke past a moving character. The potential applications are endless. Our algorithms are based on the physical equations of fluid flow, namely the Navier-Stokes equations. These equations are notoriously hard to solve when strict physical accuracy is of prime importance. Our solvers on the other hand are geared towards visual quality. Our emphasis is on stability and speed, which means that our simulations can be advanced with arbitrary time steps. We also demonstrate that our solvers are easy to code by providing a complete C code implementation in this paper. Our algorithms run in real-time for reasonable grid sizes in both two and three dimensions on standard PC hardware, as demonstrated during the presentation of this paper at the conference.



Navior-Stoke equation

diffusion of "momentum"

$$(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

convection diffusion equation

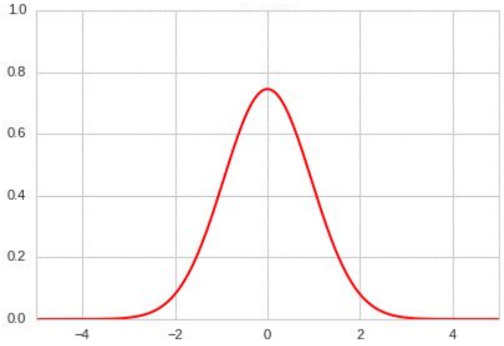
$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$

- linear
  - U= constant
- non-linear
  - U=f(x,t): Burger's equation

the good guy: diffusion/conduction

$$\frac{\partial f}{\partial t} + \cancel{U\frac{\partial f}{\partial x}} = D\frac{\partial^2 f}{\partial x^2}$$

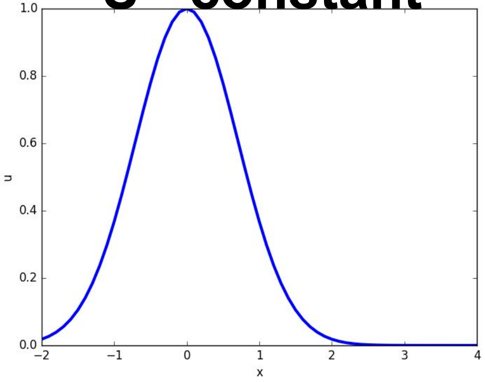
D= constant



the “bad” guy: convection/advection

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \cancel{D\frac{\partial^2 f}{\partial x^2}}$$

U= constant

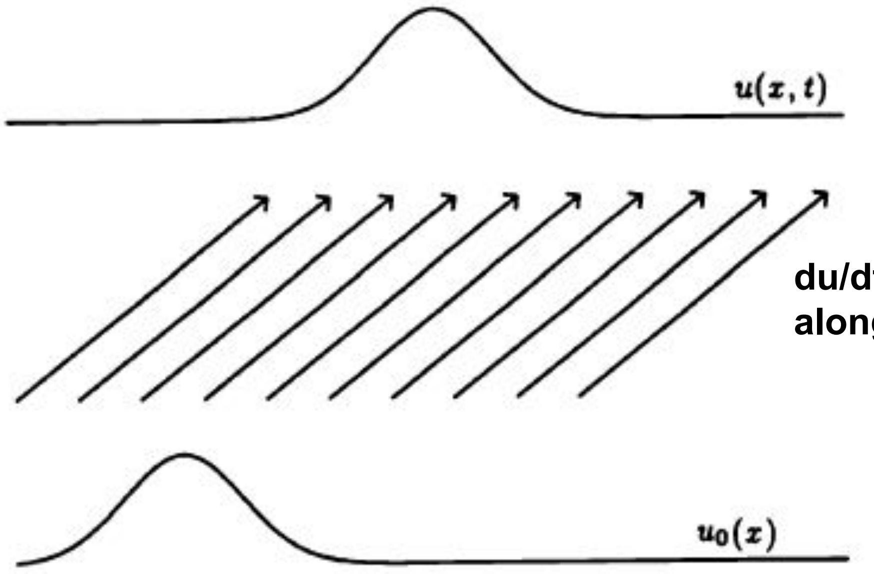


method of characteristic

see, e.g., The physics of astrophysics volume II: **gas dynamics** by Shu

linear advection

U= constant

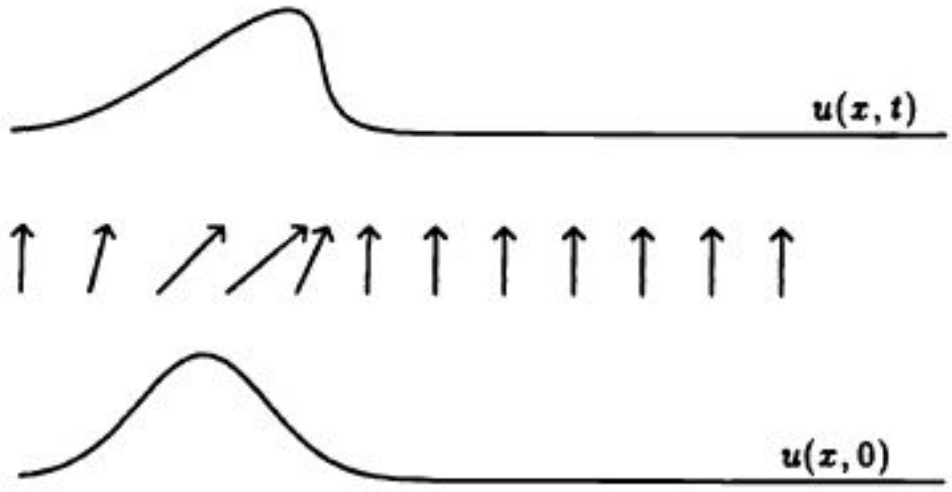


du/dt = constant
along characteristic curve

method of characteristic

U != constant

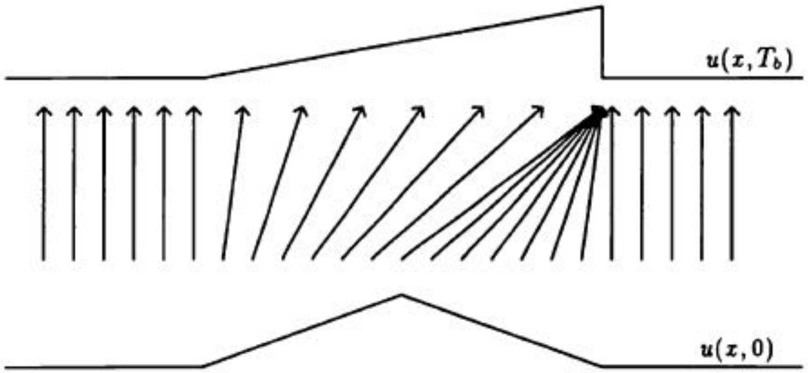
non-linear advection

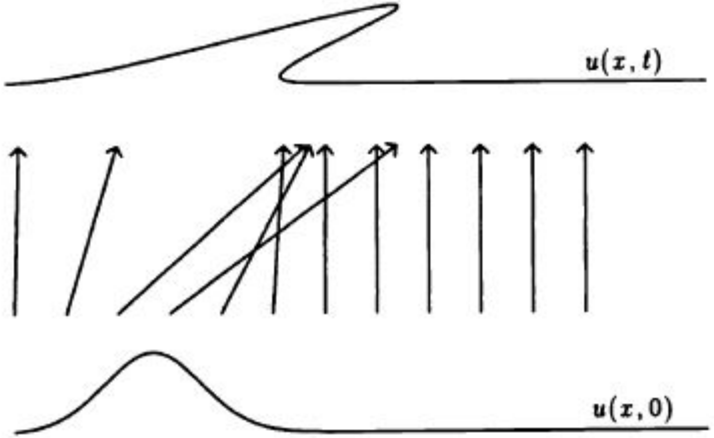


method of characteristic

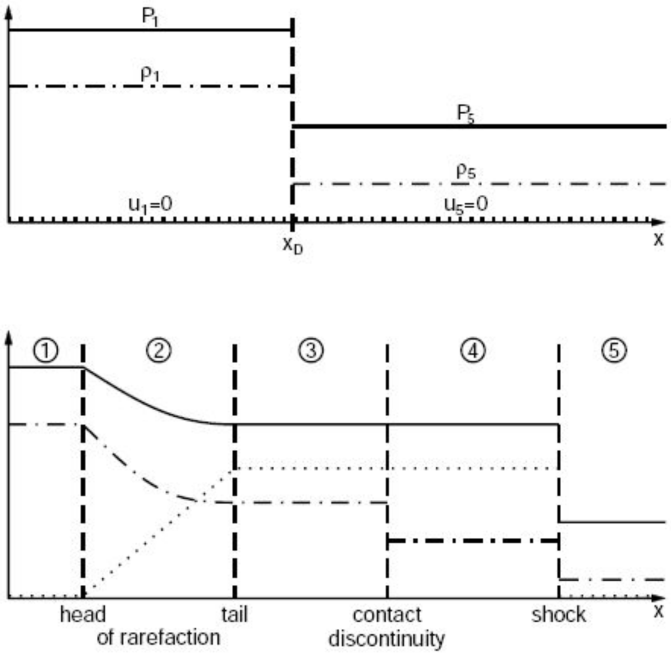
U != constant

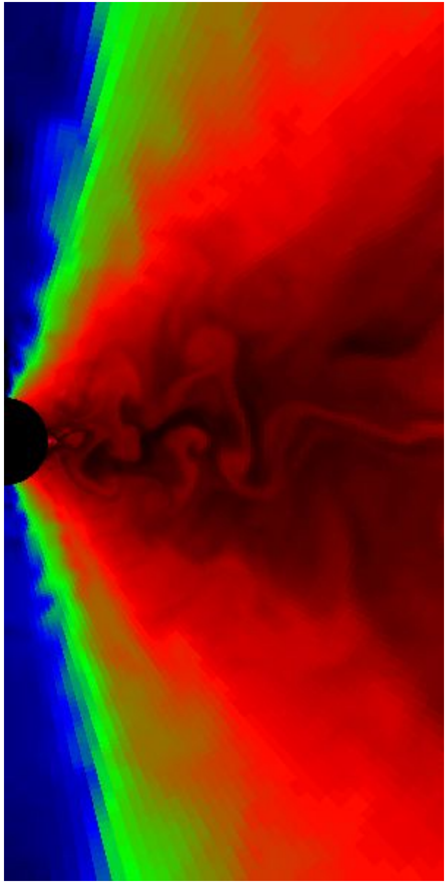
non-linear advection





Riemann problem





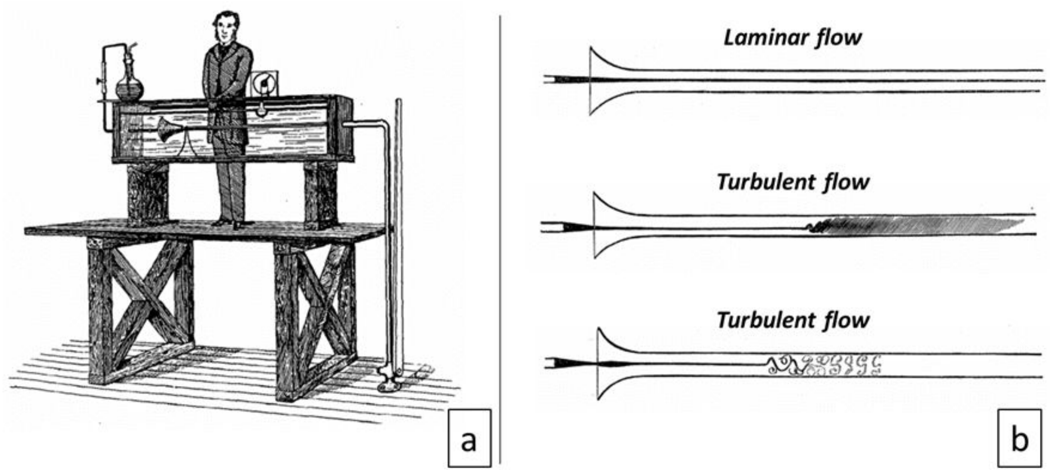
Navior-Stoke equation

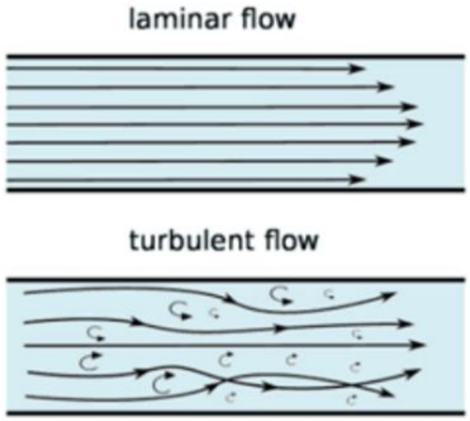
$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

turbulence appears when Reynolds number is high enough!

$$Re = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho \,.\, V \,.\, D}{\mu}$$

Rynolds' pipe experiment





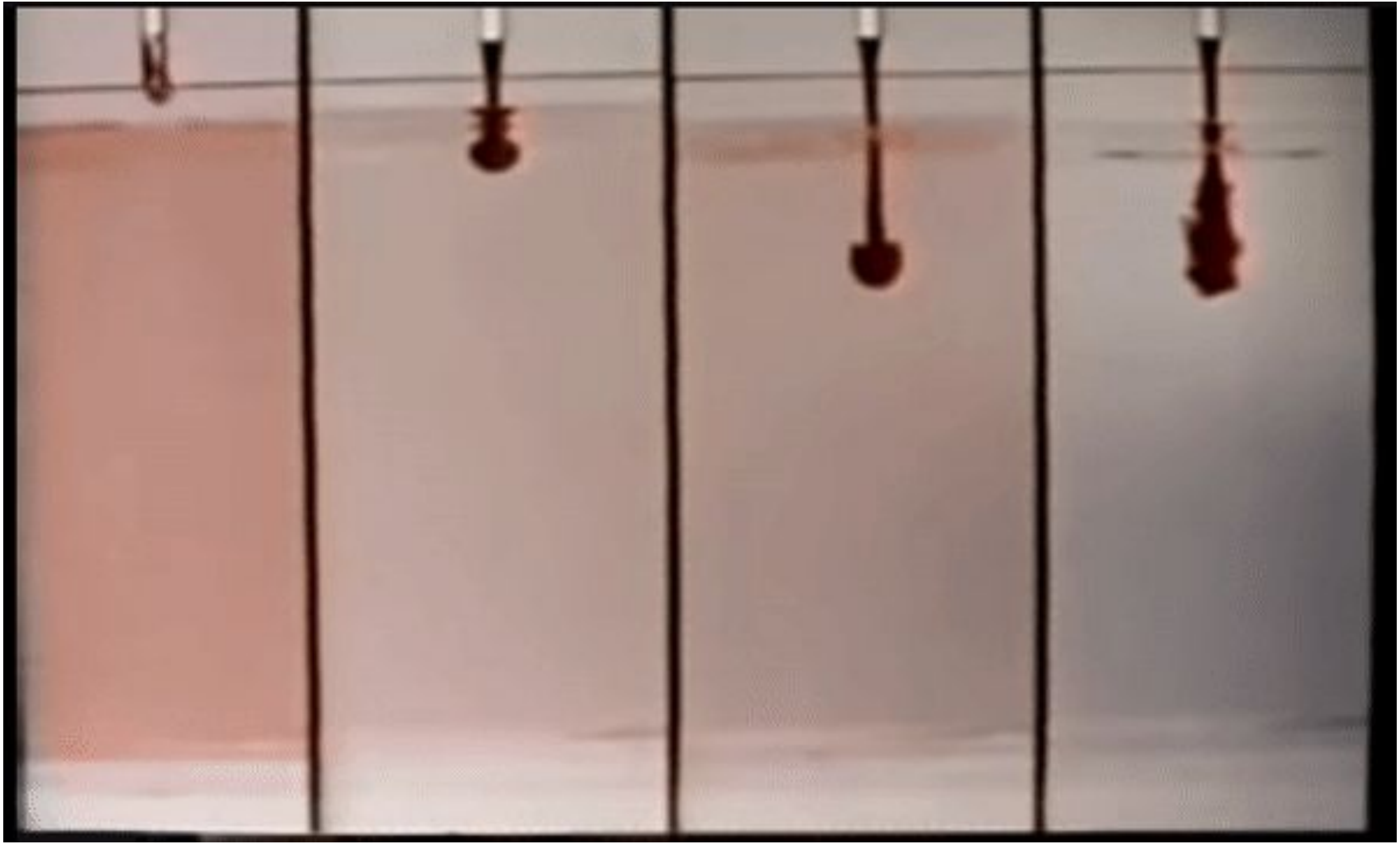
what causes turbulece?
inertial or viscosity?

inertial !

$$Re = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho \,.\, V \,.\, D}{\mu}$$

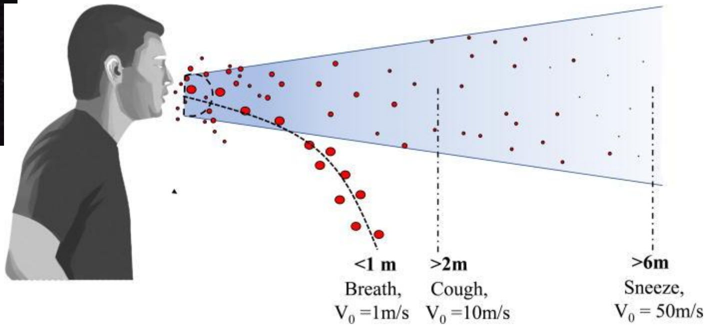
(viscousity just make you stop)

turbulensce takes place for HIGH enough Re



the ubiquitous nature of turbulence in our daily lives

Temperature (°C)	Density (kg/m^3)	Viscosity (kg/ms)
-10	1.341	1.680×10^{-5}
0	1.292	1.729×10^{-5}
10	1.246	1.778×10^{-5}
20	1.204	1.825×10^{-5}
30	1.164	1.872×10^{-5}
40	1.127	1.918×10^{-5}



acceleration ⇒ larger Re ⇒ Turbulent

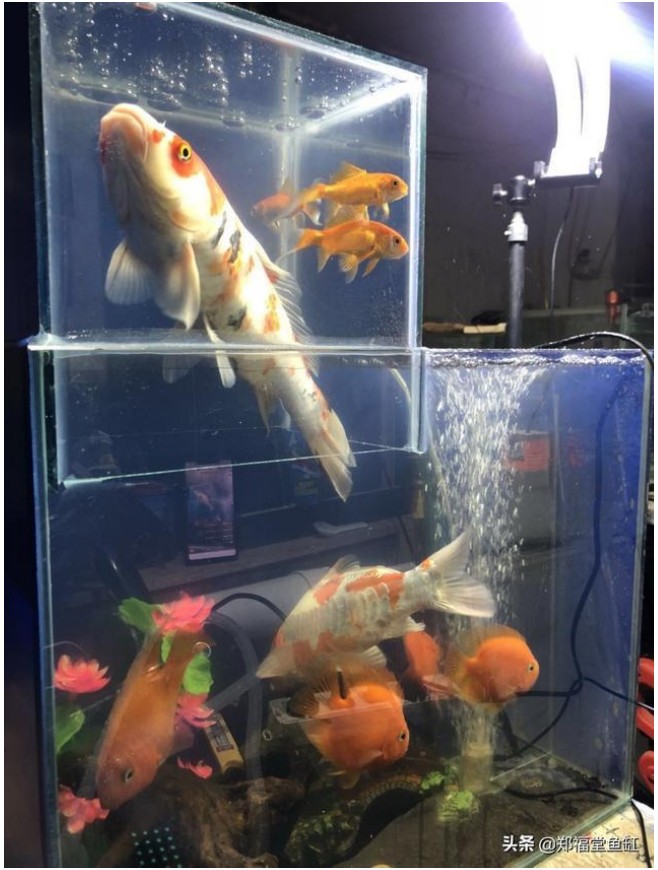
$$Re = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho \, . \, V \, . \, D}{\mu}$$

credit: Pallavi Katre et al. (Physics of Fluid)

hydrostatic: Euler equation

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

v=0, stationary, invicid



a fish tank with speicial designs

Bernoulli equation for incompressible fluid*

$$\left(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2}\vec{V}$$

stationary, invicid

integration assumption $d\rho = 0$

(incompressible)

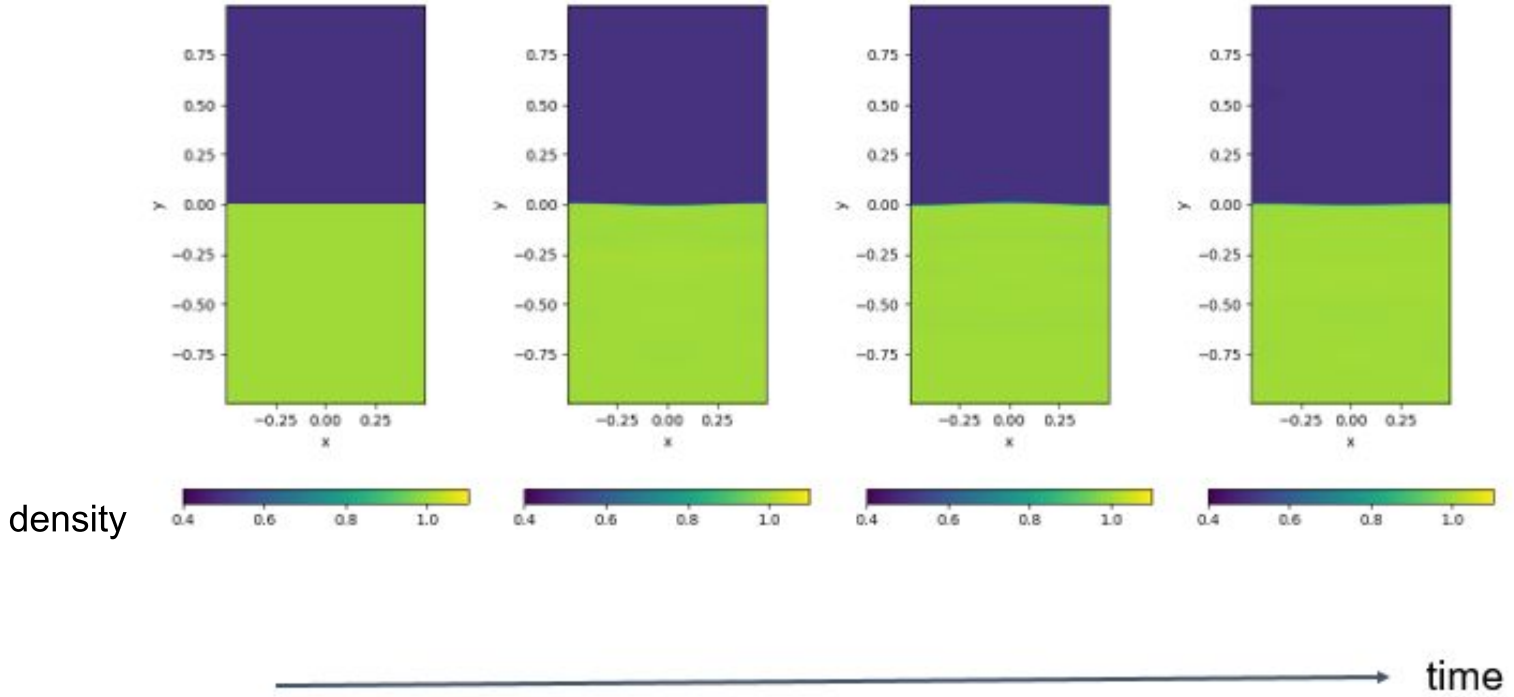
$$\boxed{\mathcal{B} = \frac{|\vec{V}|^2}{2} + \frac{P}{\rho} + gz}$$

(along a streamline)

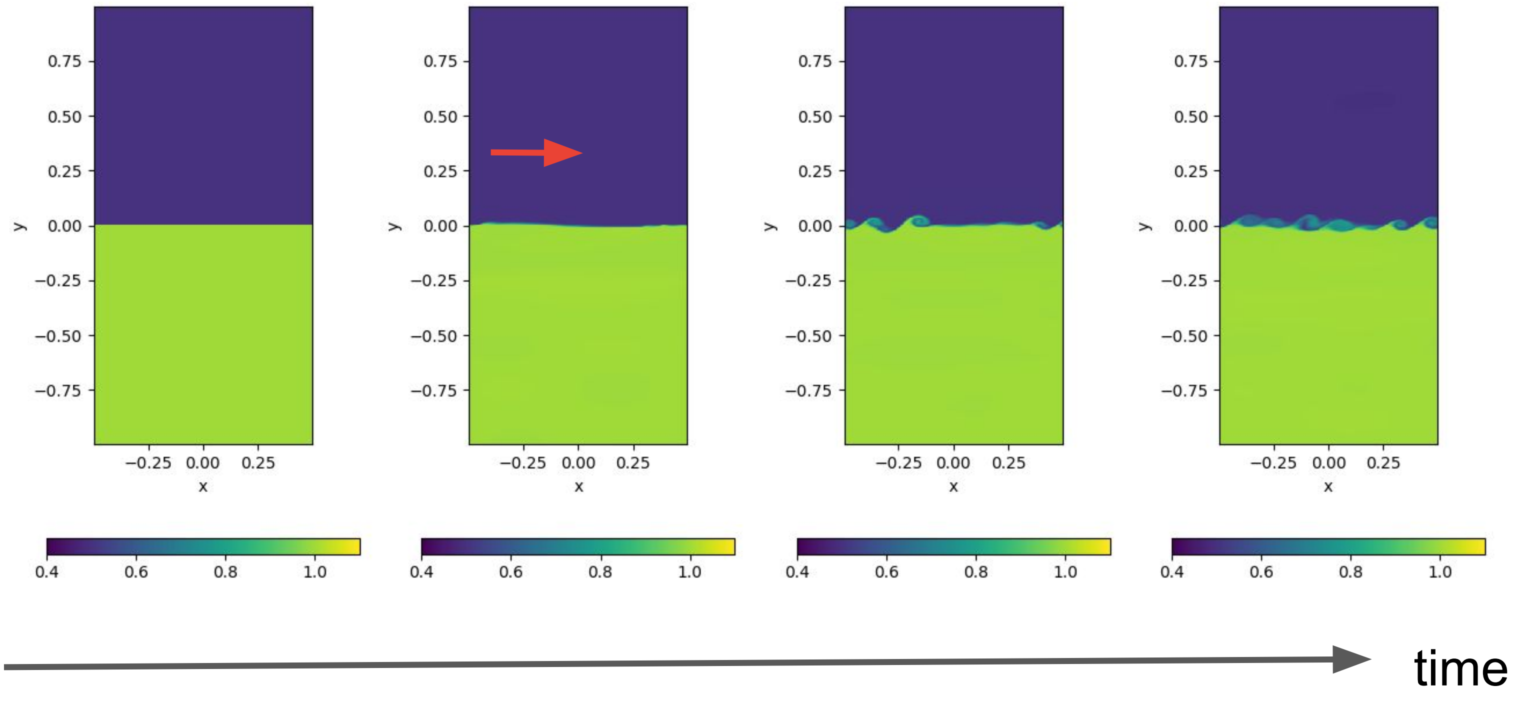
$$\mathcal{B}(x1, y1, z1) = \mathcal{B}(x2, y2, z2)$$

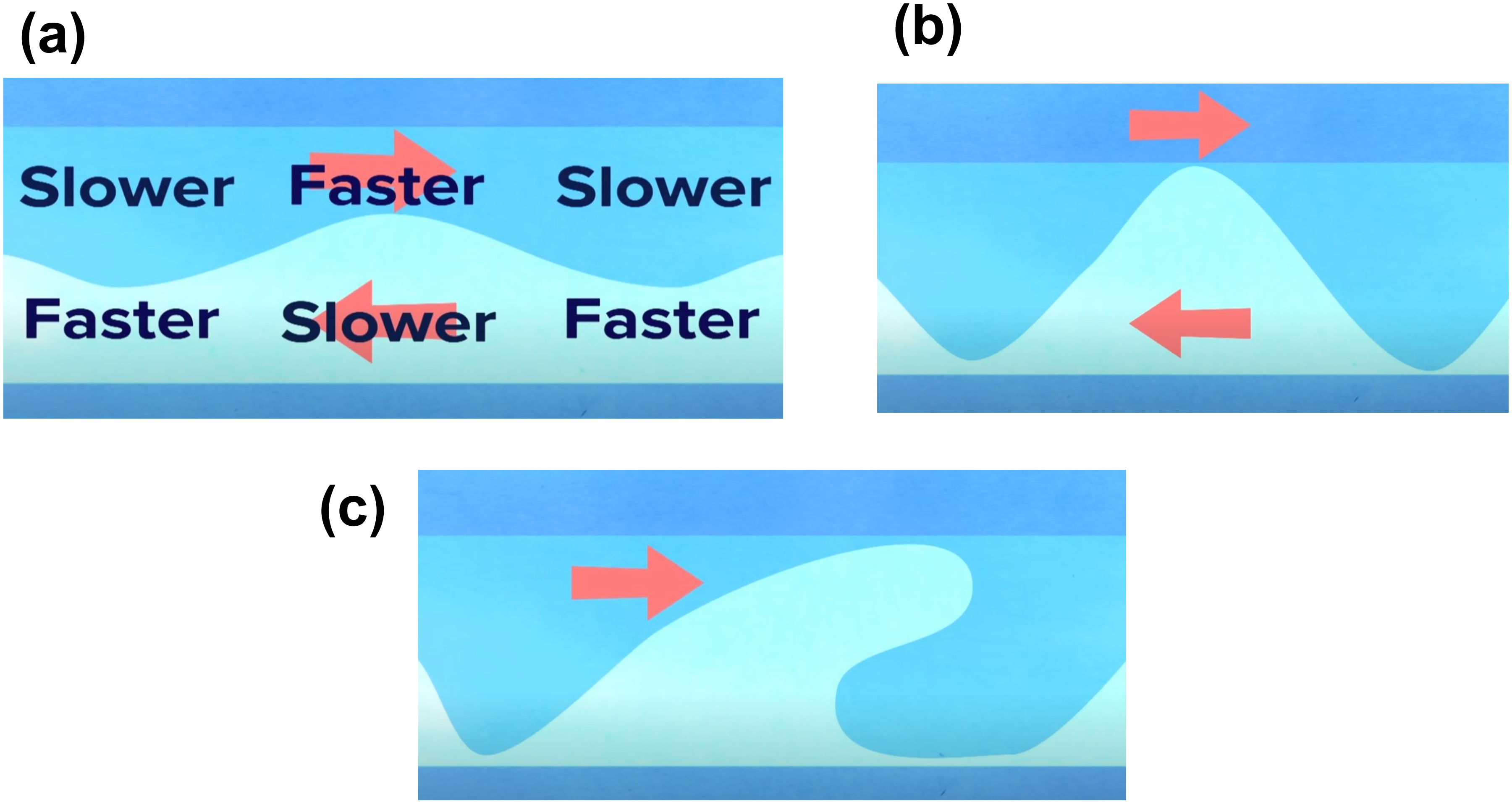
*there is Bernoulli equation for compressible fluid too! (by taking into account the change of internal energy)

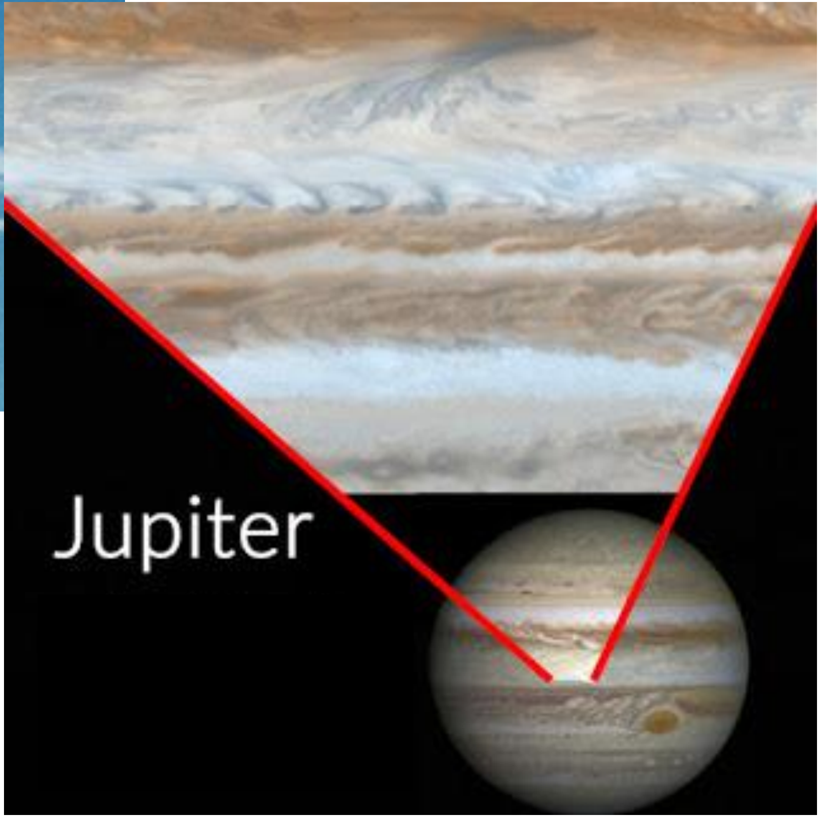
Rayleigh-Taylor stable



Kelvin-Helmholtz instability




(a)
Slower
Faster
Slower
Faster
Slower
Faster
(b)
(c)


Jupiter

governing equation III:

energy equation (1st law of thermodynamics) conservation of energy (1st law of thermodynamics)

$$\frac{D\hat{u}}{Dt} = \frac{dQ}{dt} - \frac{dW}{dt}$$

recall:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

$$p\frac{dV}{dt} = p\frac{d(\frac{1}{\rho})}{dt} = -\frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\rho\frac{d\hat{u}}{dt} + p(\nabla \cdot \vec{v}) = -\dot{Q}_{cool} + \Phi$$

Viscous dissipation function

Viscous dissipation function

$$\Phi = \mu[2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial w}{\partial z})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2] > 0$$

→ always increase the internal energy (inreversible) compressible

$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u} \\ \dfrac{D\mathbf{u}}{Dt} = -\dfrac{\nabla p}{\rho} + \mathbf{g} \\ \rho\dfrac{d\hat{u}}{dt} + p(\nabla\cdot\vec{v}) = -\dot{Q}_{cool} + \Phi \end{cases}$$

incompressible

$$\begin{cases} \dfrac{D\rho}{Dt} = 0 \\ \dfrac{D\mathbf{u}}{Dt} = -\dfrac{\nabla p}{\rho} + \mathbf{g} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

*six unknowns: density + pressure + gravitational potential + 3D velocities equation of state (EOS)

barotropic $p(\rho)$

isothermal $p \propto \rho$ $$C_s = \frac{d\rho}{dp} = \frac{\rho}{p}$$

adiabatic $p \propto \rho^\gamma$ $$C_s = \frac{d\rho}{dp} = \gamma\frac{\rho}{p}$$

*in general, EOS will not be barotropic. We need to solve energy equation which follows the heating and cooling in the gas equation of state (EOS) : adiabatic $\boldsymbol{p = k\rho^{\gamma}}$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}$$

$$d(ln\ p) = \gamma d(ln\ \rho)$$

$$\boxed{\frac{D}{Dt}(\frac{p}{\rho^{\gamma}}) = 0}$$

cf. incompressible

$$\frac{D\rho}{Dt} = 0$$

example: the importance of energy equation

$$\dot{Q}^{+} \approx \dot{Q}^{-} (\gg \dot{Q}^{adv})$$

(radiative efficient)

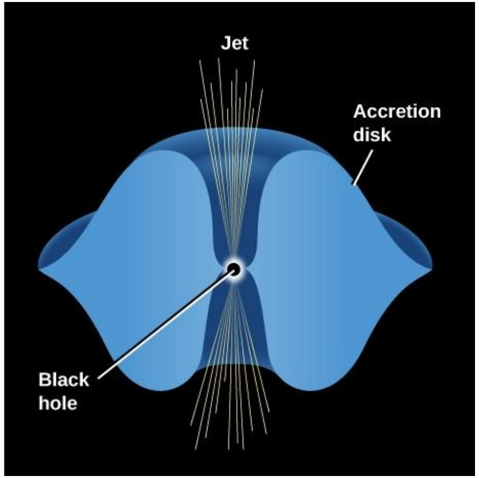
$$TdS = dQ$$

$$\rho T \frac{ds}{dt} = \dot{Q}^{+} - \dot{Q}^{-}$$

$$\boxed{\rho v_r T \frac{ds}{dr} \equiv \dot{Q}^{adv} = \dot{Q}^{+} - \dot{Q}^{-}}$$

(radiative inefficient)

$$\dot{Q}^{+} \approx \dot{Q}^{adv} (\gg \dot{Q}^{-})$$



conservative form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = J$$

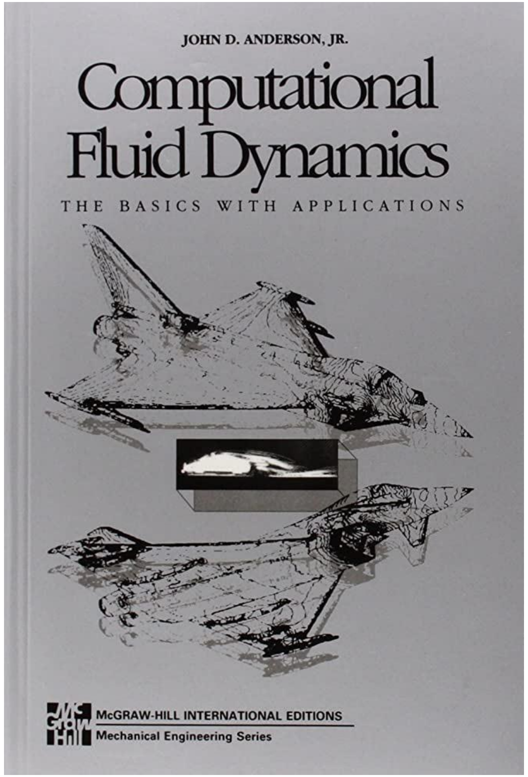
$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho\left(e + \frac{V^2}{2}\right) \end{Bmatrix}$$

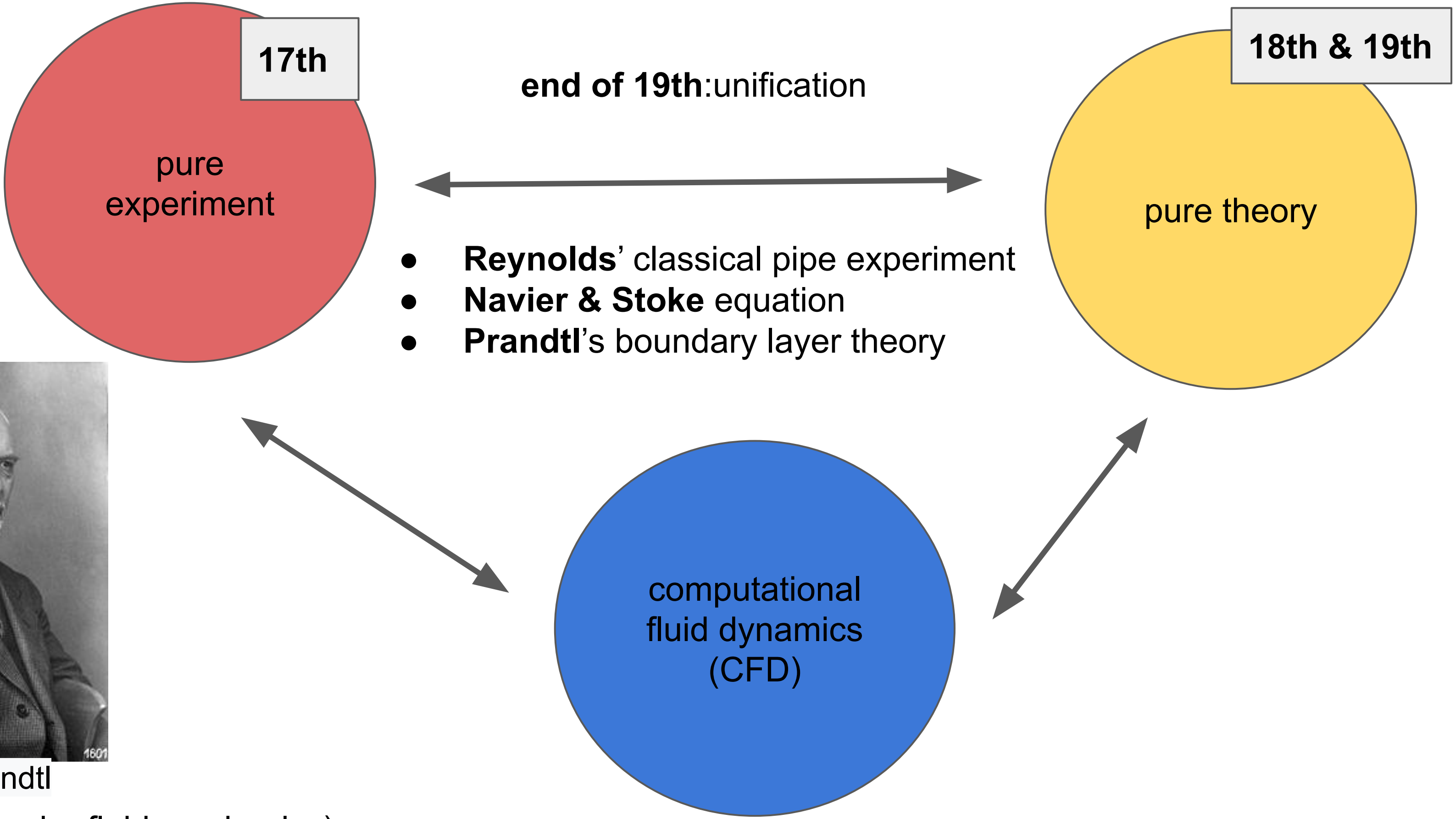
$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho v u - \tau_{xy} \\ \rho w u - \tau_{xz} \\ \rho\left(e + \frac{V^2}{2}\right)u + pu - k\frac{\partial T}{\partial x} - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} \end{Bmatrix}$$

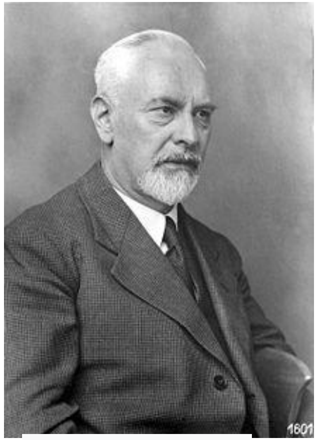
$$G = \begin{Bmatrix} \rho v \\ \rho u v - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho w v - \tau_{yz} \\ \rho\left(e + \frac{V^2}{2}\right)w + pv - k\frac{\partial T}{\partial y} - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} \end{Bmatrix}$$

$$H = \begin{Bmatrix} \rho w \\ \rho u w - \tau_{zx} \\ \rho v w - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ \rho\left(e + \frac{V^2}{2}\right)w + pw - k\frac{\partial T}{\partial z} - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} \end{Bmatrix}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(u f_x + v f_y + w f_z) + \rho\dot{q} \end{Bmatrix}$$



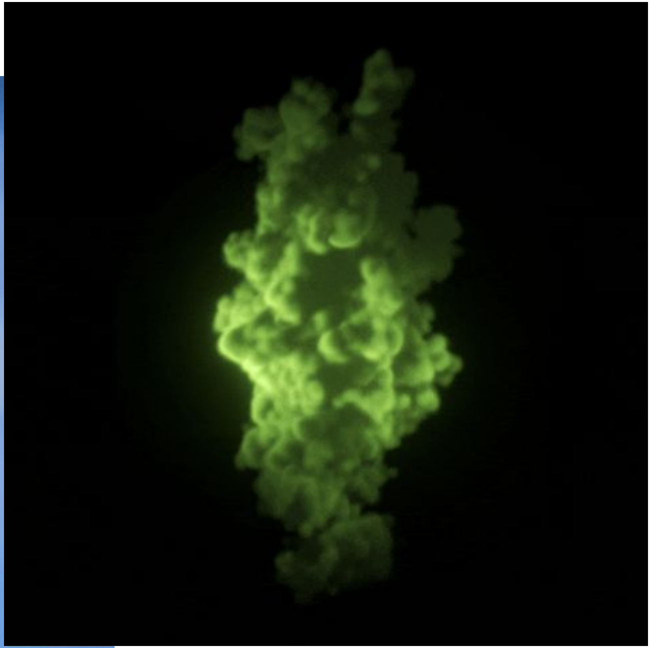




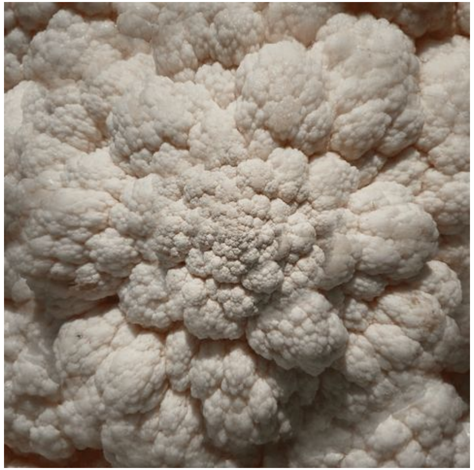
Ludwig Prandtl

(father of moder fluid mechanics)

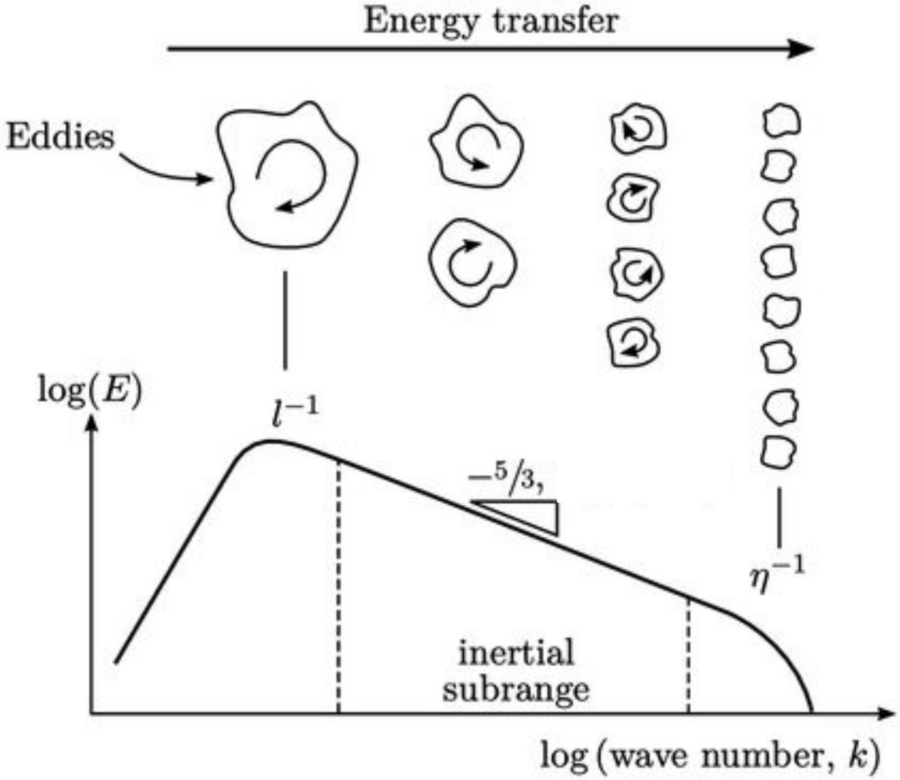
understanding turbulent flows….



self-simliar



key word: eddies, energy cascade

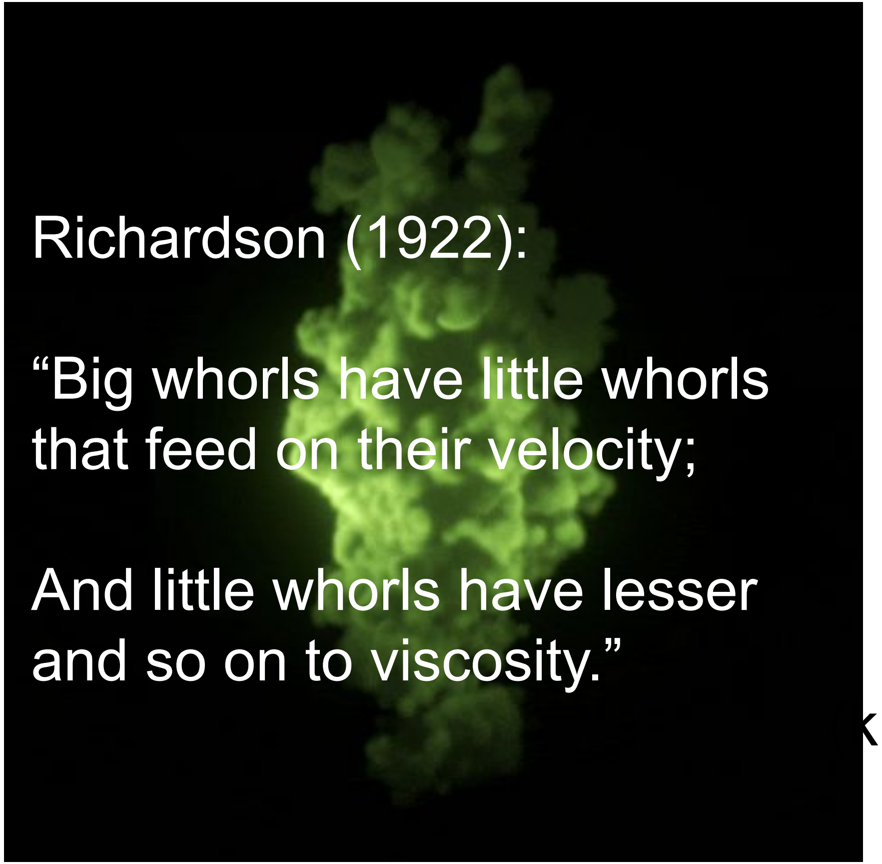


laminar flow

turbulent flow

deformation
→ viscous dissipation

(kinetic) energy cascade to smaller eddies
→ viscous dissipation



Richardson (1922):

“Big whorls have little whorls
that feed on their velocity;

And little whorls have lesser
and so on to viscosity.”

turbulent flow

kinetic) energy cascade to smaller
eddies
→ viscous dissipation

Komogrolov's -5/3 law

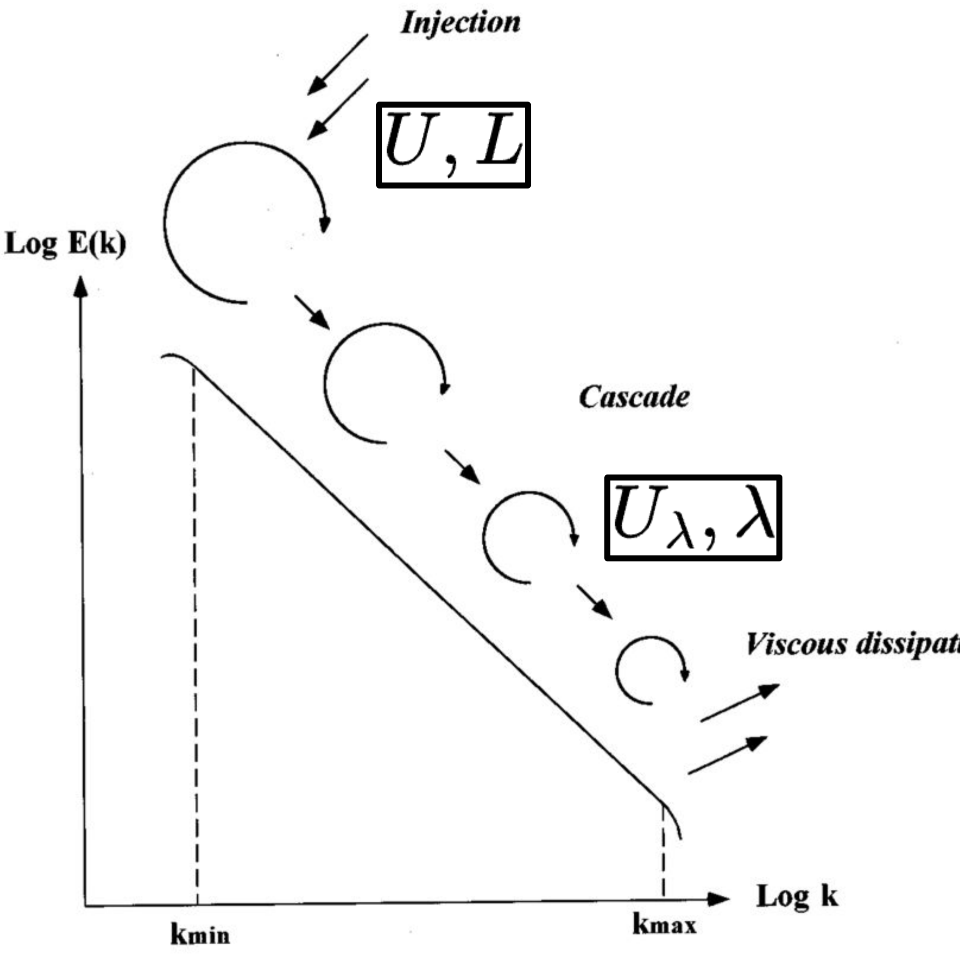
homogeneous and isotropic

$$\epsilon = \frac{E}{t} = \frac{U^2}{L/U} = \frac{U_\lambda^2}{\lambda/U_\lambda}$$

$$\Rightarrow U_\lambda = U(\frac{\lambda}{L})^{1/3}$$

$$\Rightarrow E(k) \propto U_\lambda^2 \propto \lambda^{2/3} \propto k^{-2/3}$$

$$\Rightarrow E(k)dk = k^{-5/3}dk$$

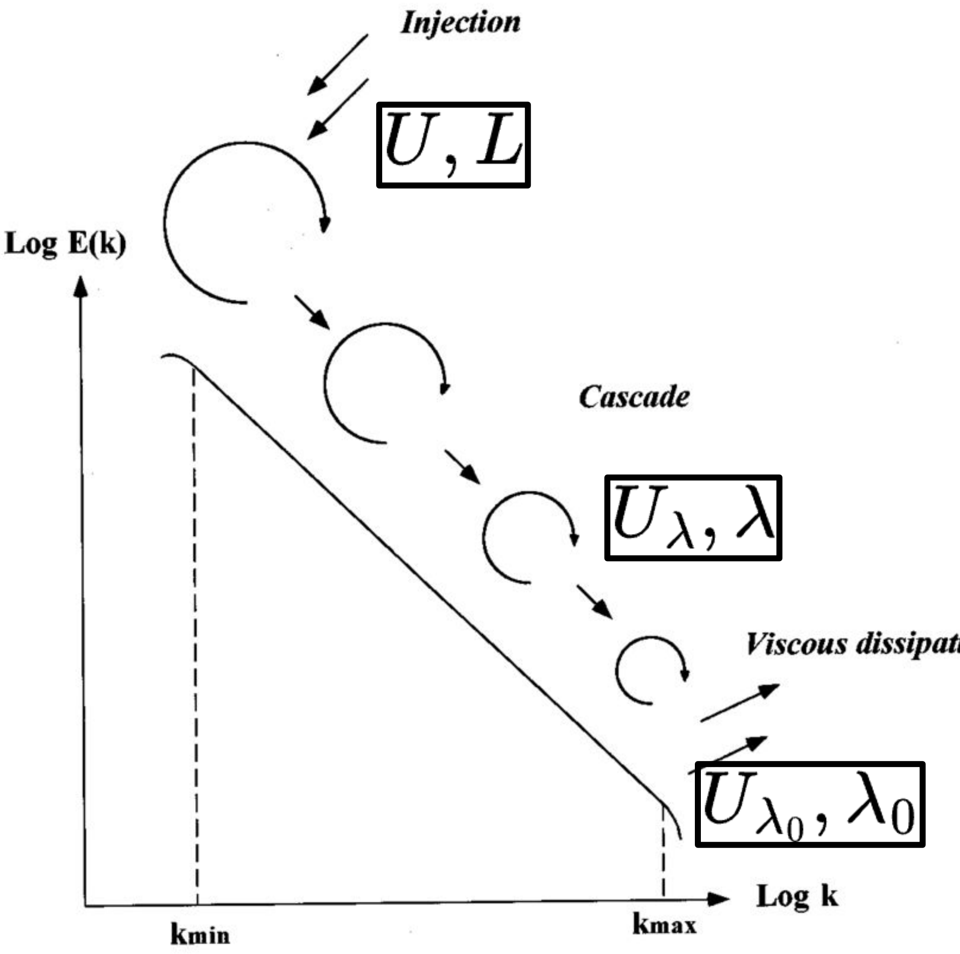


recall: Viscous dissipation function

$$\Phi = \mu[2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial w}{\partial z})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2]$$

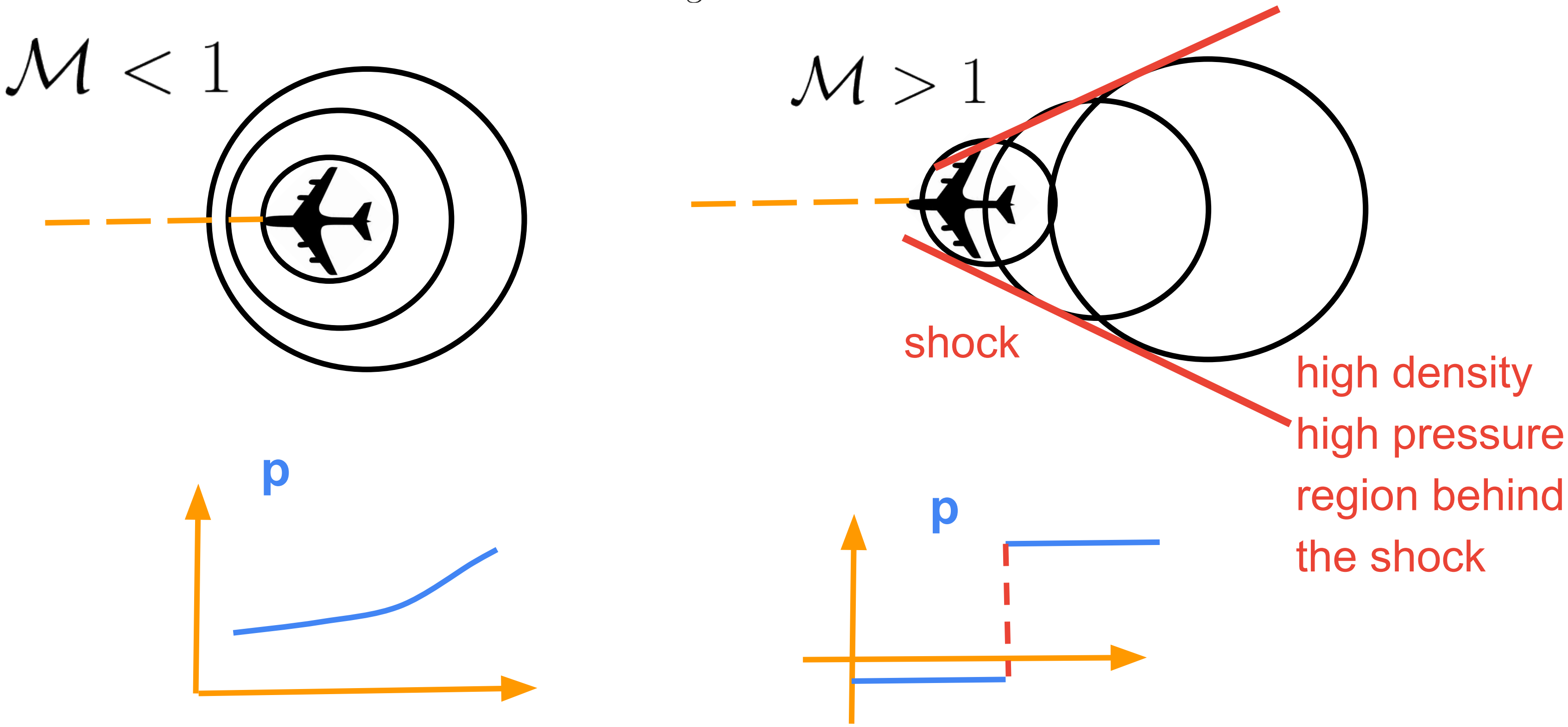
$$\frac{U_{\lambda_0}^2}{\lambda_0/U_{\lambda_0}} \sim \nu(\frac{U_{\lambda_0}}{\lambda_0})^2$$

$$\Longrightarrow \lambda_0 \sim R_e^{-3/4} L$$



what is the condition for treating (compressible) gas as incompressible?

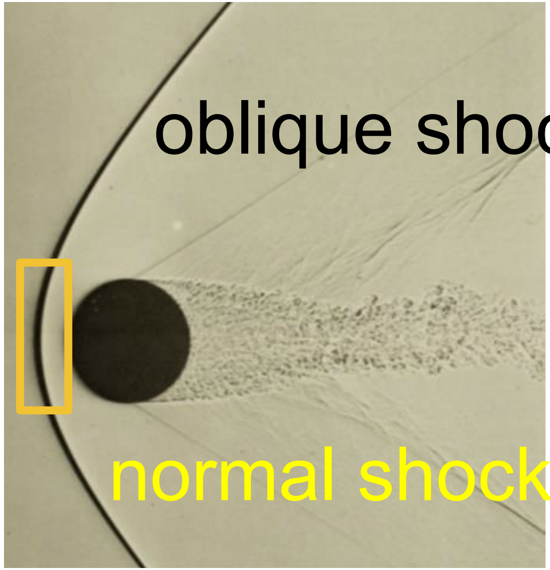
Mach number: $$\mathcal{M} \equiv \frac{V}{C_s}$$



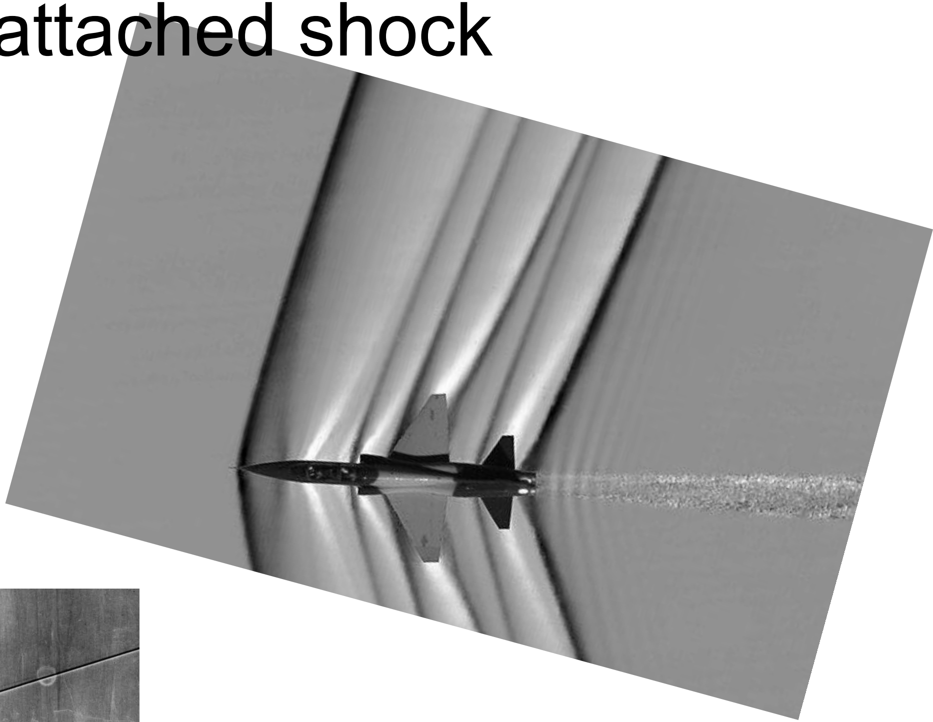
(at the flight-nose frame)

what one is Mach cone?

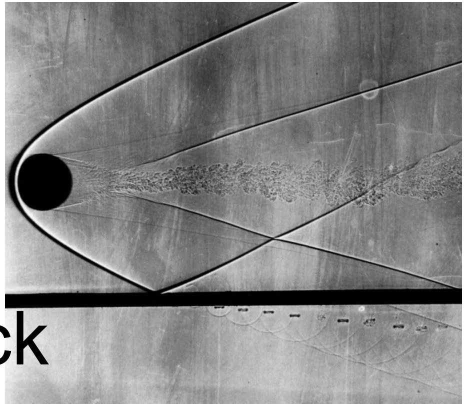
detached shock



attached shock



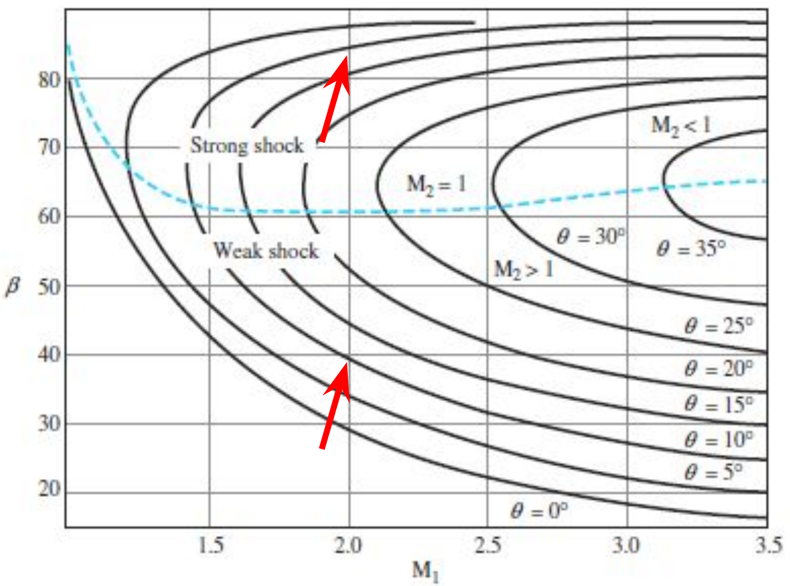
avoiding bow shock (and therefore drag)

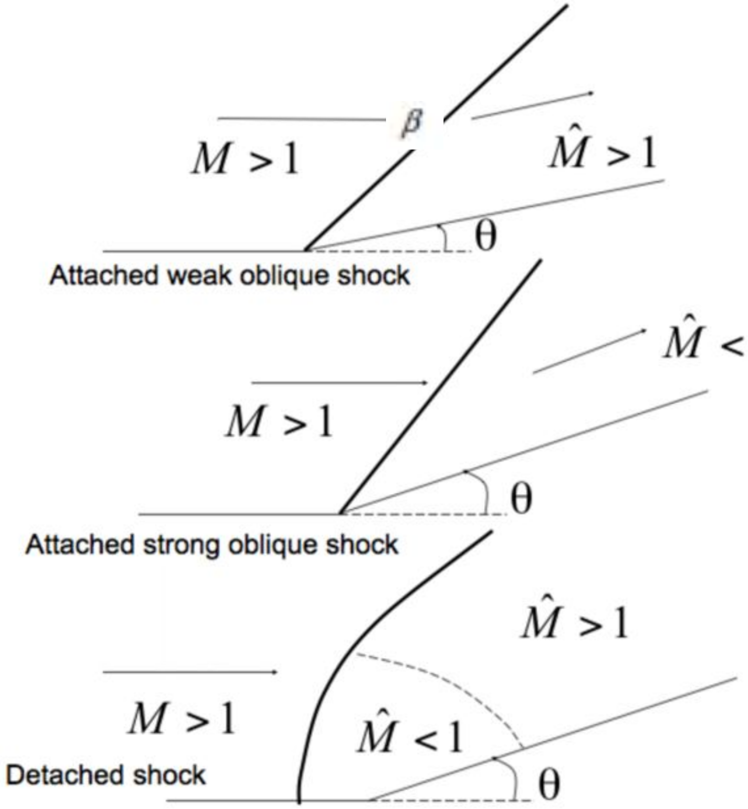


reflection of shock

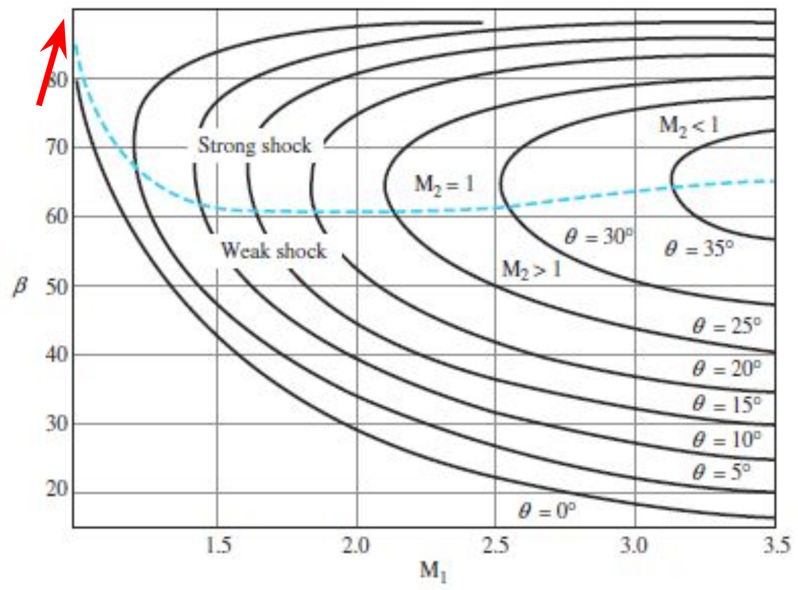
M_1=2 defelction angle=10 degree

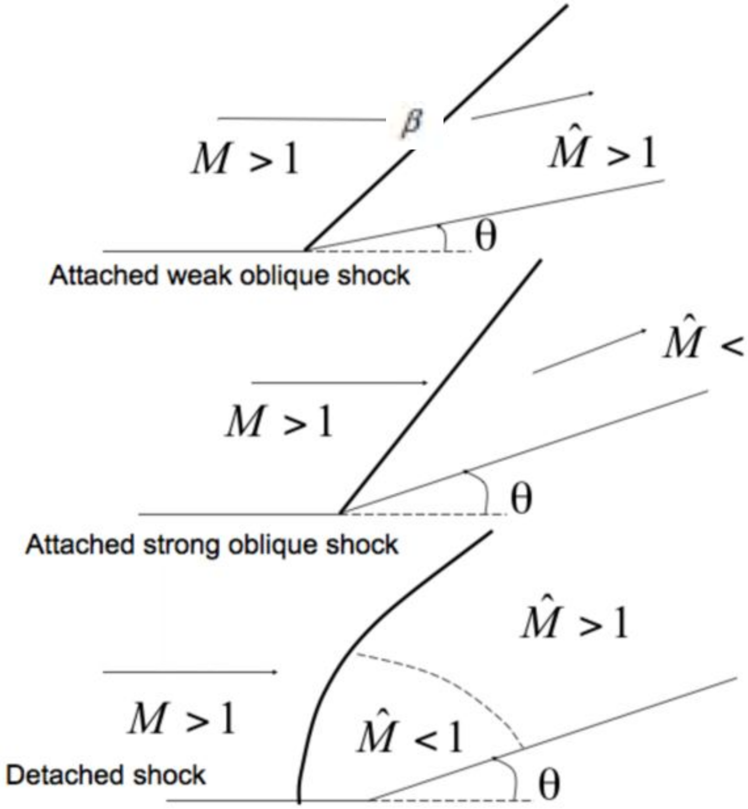
⇒ shock wave angle = 40 degree or   85 degree



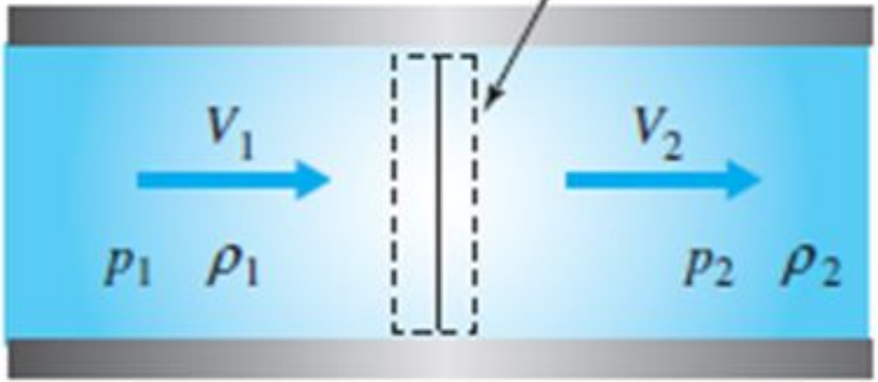


normal shock: $\beta = 90^\circ$



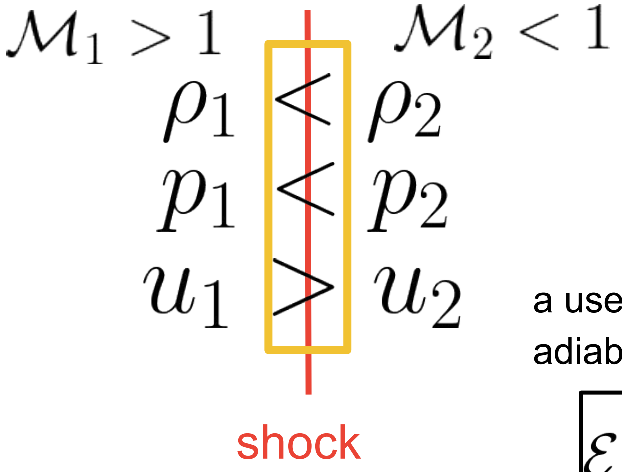


normal shock: must be a strong shock (M_2<1)



- fluid assumption failed (Beurnolli equation cannot be applied)
- kinetic energy convert to heat energy

(adiabatic) **normal shock** and Rankine-Hugoniot relations



RH relation at the shock frame $(\frac{\partial}{\partial t} = 0)$:

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ \frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2} \end{cases}$$

a useful relation for adiabatic flow:

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}}$$

if $\mathcal{M}_1 \to \infty$

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma + 1}{\gamma - 1}$$

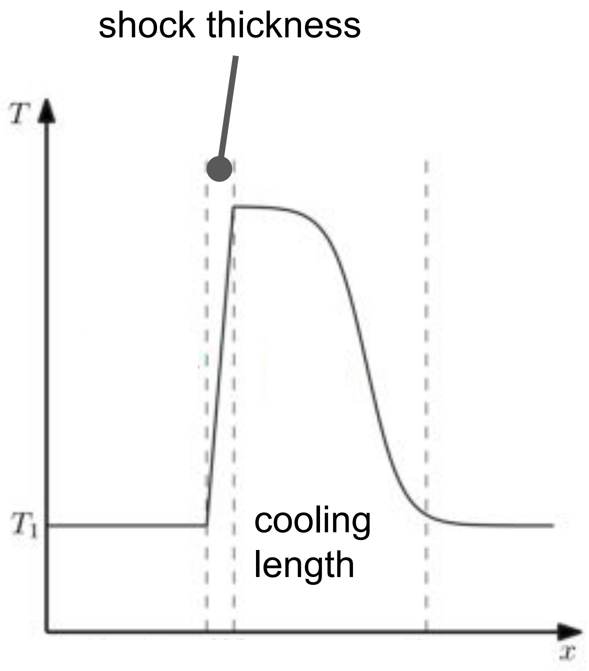
For γ=5/3, strong shocks have $\rho_2/\rho_1 = 4$

*for isothermal or radiative shock, density contrast can reach arbitrarily high values

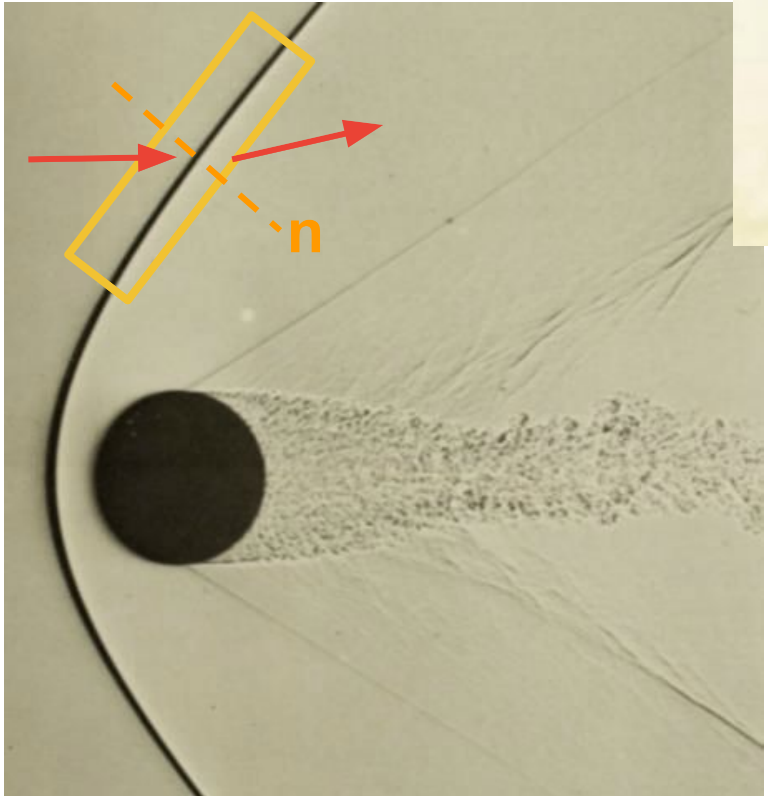
(isothermal) normal shock

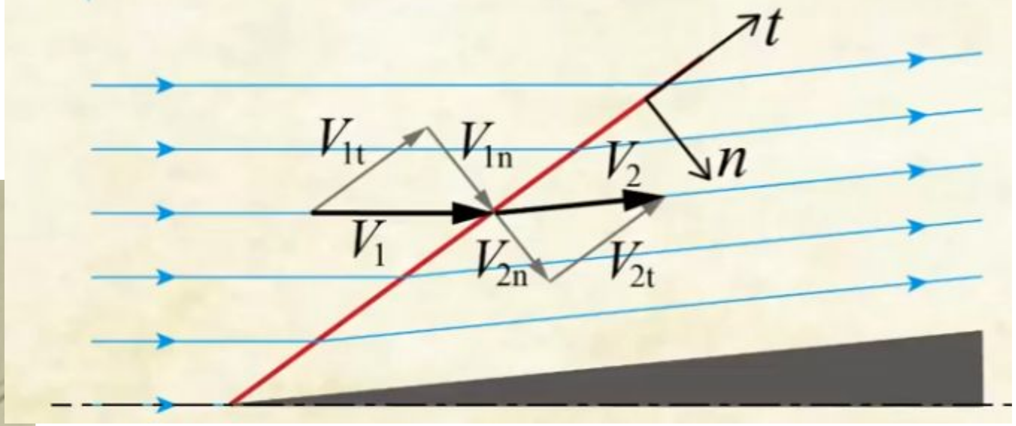
RH relation at the shock frame $(\frac{\partial}{\partial t} = 0)$:

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ T_1 = T_2 \end{cases}$$



oblique shock



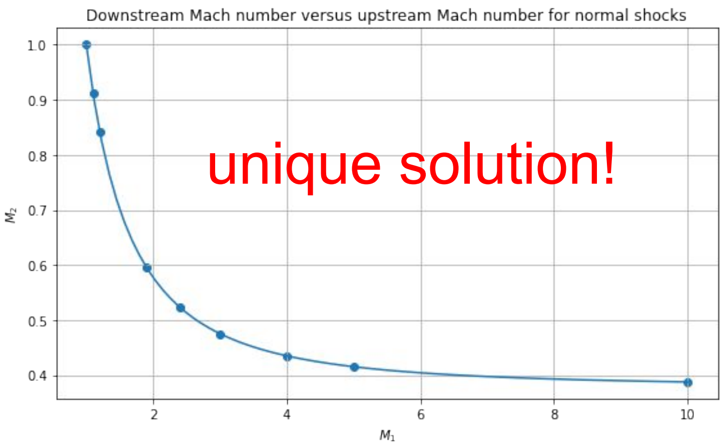


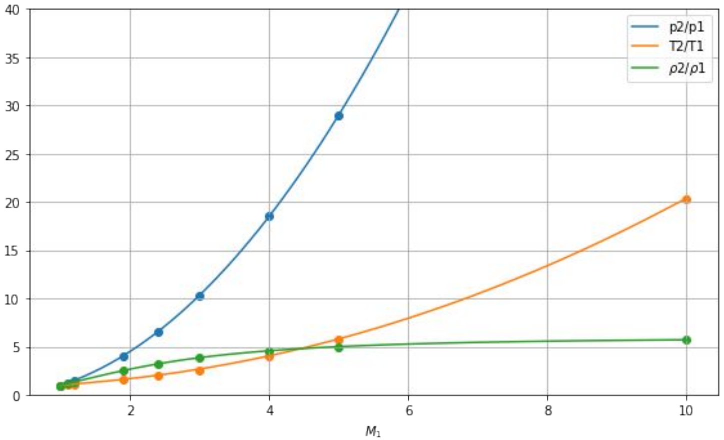
$$\rho_1 u_{1\mathbf{n}} = \rho_2 u_{2\mathbf{n}}$$

$$\rho_1 u_{1\mathbf{n}}^2 + p_1 = \rho_2 u_{2\mathbf{n}}^2 + p_2$$

The (adiabatic) normal shock tables (for air;r γ = 1.4)

M_i	M_e	$\frac{p_e}{p_i}$	$\frac{\rho_e}{\rho_i}$	$\frac{T_e}{T_i}$	$\frac{p_{t,e}}{p_{t,i}}$	$\frac{A_{*,e}}{A_{*,i}}$	$\frac{V_e}{V_i}$
1.00	1.00000	1.0000	1.0000	1.0000	1.00000	1.00000	1.0000
1.10	0.91177	1.2450	1.1691	1.0649	0.99892	1.00107	0.8554
1.20	0.84217	1.5133	1.3416	1.1280	0.99280	1.00725	0.7453
1.30	0.78596	1.8050	1.5157	1.1909	0.97935	1.02106	0.6597
1.40	0.73971	2.1200	1.6896	1.2547	0.95819	1.04364	0.5918
1.50	0.70109	2.4583	1.8621	1.3202	0.92978	1.07553	0.5370
1.60	0.66844	2.8200	2.0317	1.3880	0.89520	1.11709	0.4921
1.70	0.64055	3.2050	2.1977	1.4583	0.85573	1.16864	0.4550
1.80	0.61650	3.6133	2.3592	1.5316	0.81268	1.23054	0.4238
1.90	0.59562	4.0450	2.5157	1.6079	0.76735	1.30325	0.3974
2.00	0.57735	4.5000	2.6666	1.6875	0.72088	1.38732	0.3749
2.10	0.56128	4.9784	2.8119	1.7704	0.67422	1.48338	0.3556
2.20	0.54706	5.4800	2.9512	1.8569	0.62812	1.59221	0.3388
2.30	0.53441	6.0050	3.0846	1.9468	0.58331	1.71466	0.3241
2.40	0.52312	6.5533	3.2119	2.0403	0.54015	1.85170	0.3113
2.50	0.51299	7.1250	3.3333	2.1375	0.49902	2.00438	0.2999
2.60	0.50387	7.7200	3.4489	2.2383	0.46012	2.17387	0.2899
2.70	0.49563	8.3383	3.5590	2.3429	0.42359	2.36144	0.2809
2.80	0.48817	8.9800	3.6635	2.4512	0.38946	2.56846	0.2729
2.90	0.48138	9.6450	3.7629	2.5632	0.35773	2.79639	0.2657
3.00	0.47519	10.333	3.8571	2.6790	0.32834	3.04681	0.2592
4.00	0.43496	18.500	4.5714	4.0469	0.13876	7.21309	0.2187
5.00	0.41523	29.000	5.0000	5.8000	0.06172	16.22510	0.1999
10.00	0.38757	116.50	5.7143	20.388	0.00304	329.56100	0.1749
∞	0.37796	∞	6.000	∞	0	∞	0.1667





$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

$$\rho u A = \text{constant}$$

$$u\,du + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$\rightarrow \quad \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

$$\boxed{(1-\mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

$$\rho u A = \text{constant}$$

$$u\,du + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$\rightarrow \quad \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

<1%   0.3   10%

$$\boxed{(1 - \mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

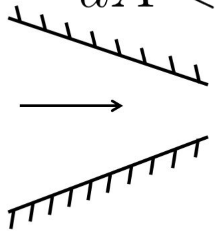
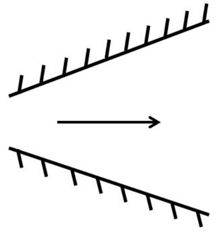
if $\mathcal{M} < 0.3 \;\rightarrow$ impcompressible fluid incompressible:

$$\nabla \cdot \mathbf{v} = 0$$

or

$$\mathcal{M} < 0.3$$

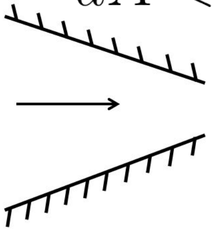
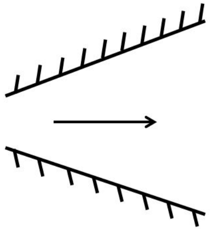
$$(1 - \mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}$$

	subsonic $1 - \mathcal{M}^2 > 0$	supersonic $1 - \mathcal{M}^2 < 0$
$dA < 0$ 	$du > 0$	$du < 0$
$dA > 0$ 	$du < 0$	$du > 0$

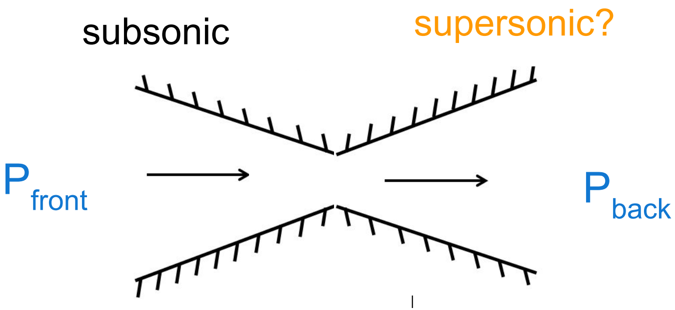
$$(1-\mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}$$

why?

$$\rho u A = \text{constant}$$

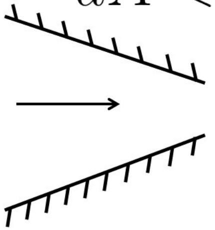
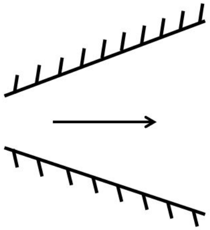
	subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$ 	$du>0$	$du<0$
$dA>0$ 	$du<0$	$du>0$

(convergence-divergence) nozzle flow



- A supersonic flow can be produced when the back pressure (P_{back}) is low enough

- If the sonic transition does no occur in the nozzle flow, the fluid speed reaches an extremum (du=0) when dA=0

	subsonic $1 - \mathcal{M}^2 > 0$	supersonic $1 - \mathcal{M}^2 < 0$
$dA < 0$ 	$du > 0$	$du < 0$
$dA > 0$ 	$du < 0$	$du > 0$

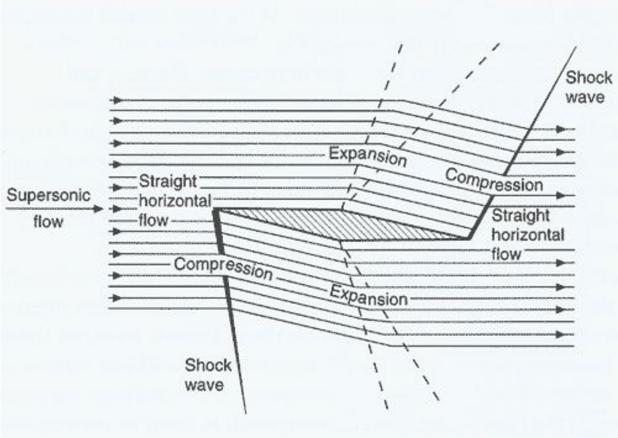
for supersonic flow…

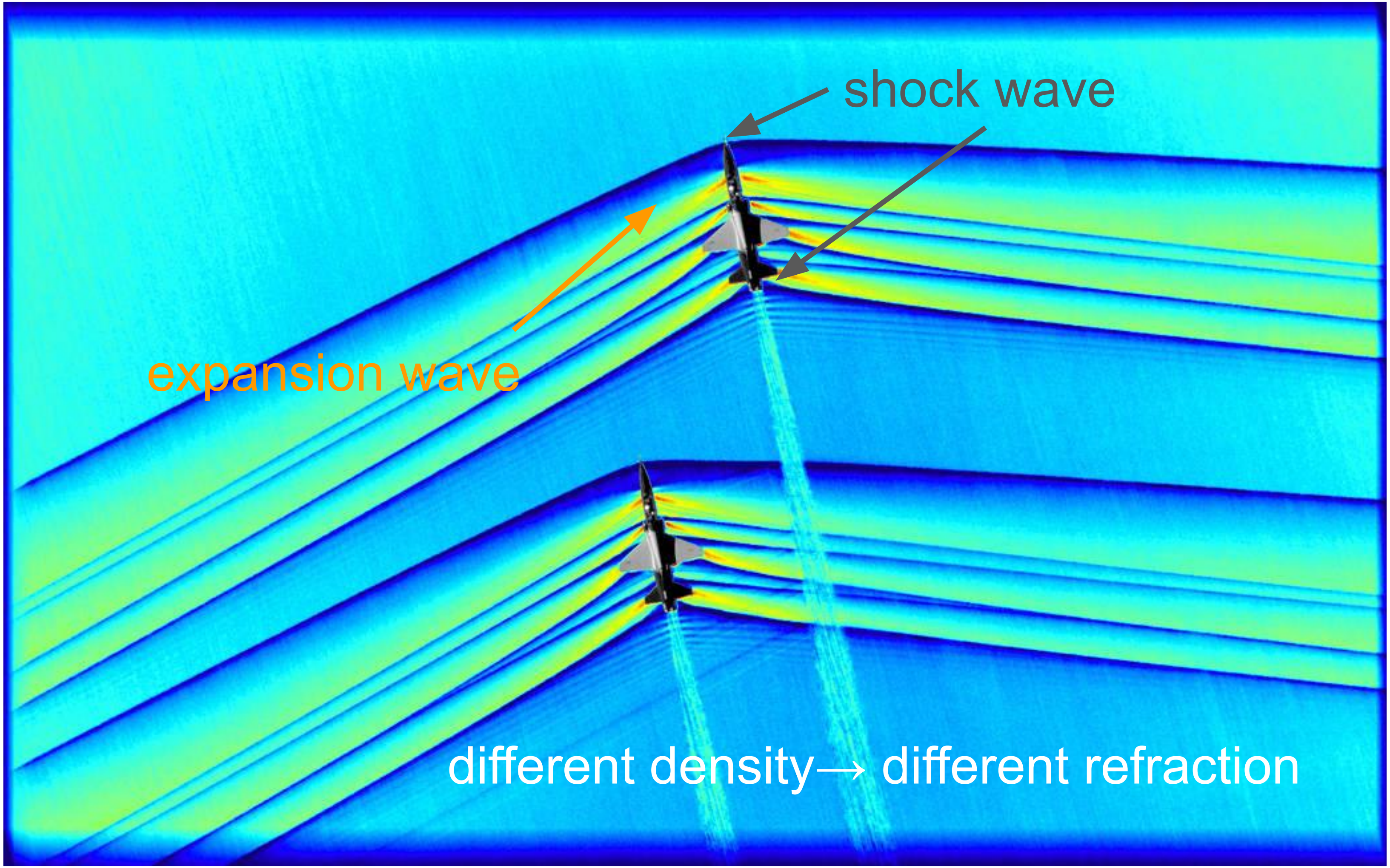
compression (Temp. increases):

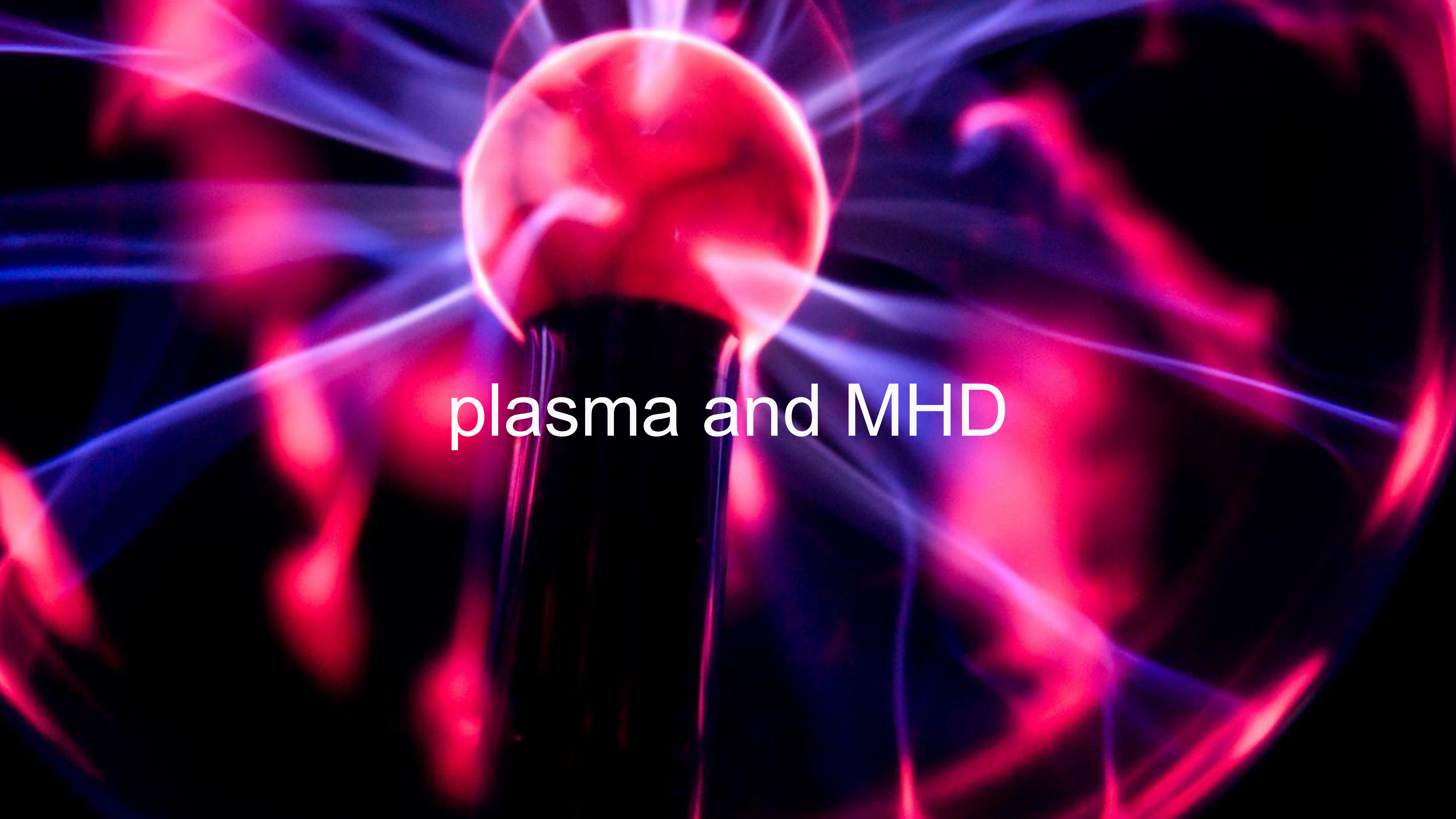
pressure (sudden) increases when flow pass inward corner, v decreases

expansion (**Temp. decreases**):

pressure (sudden) decreses when flow pass outward corner, v increases



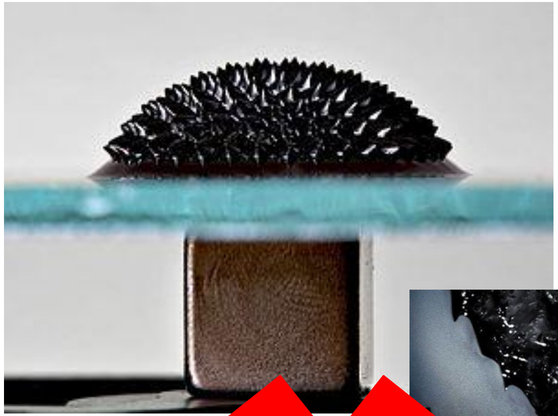

shock wave
expansion wave
different density→ different refraction



plasma and MHD

MagnetoHydroDynamics（磁流體力學）

credit: wiki



we are NOT talking about

ferrofluid (鐵磁流體)

before MHD: paritcle orbits

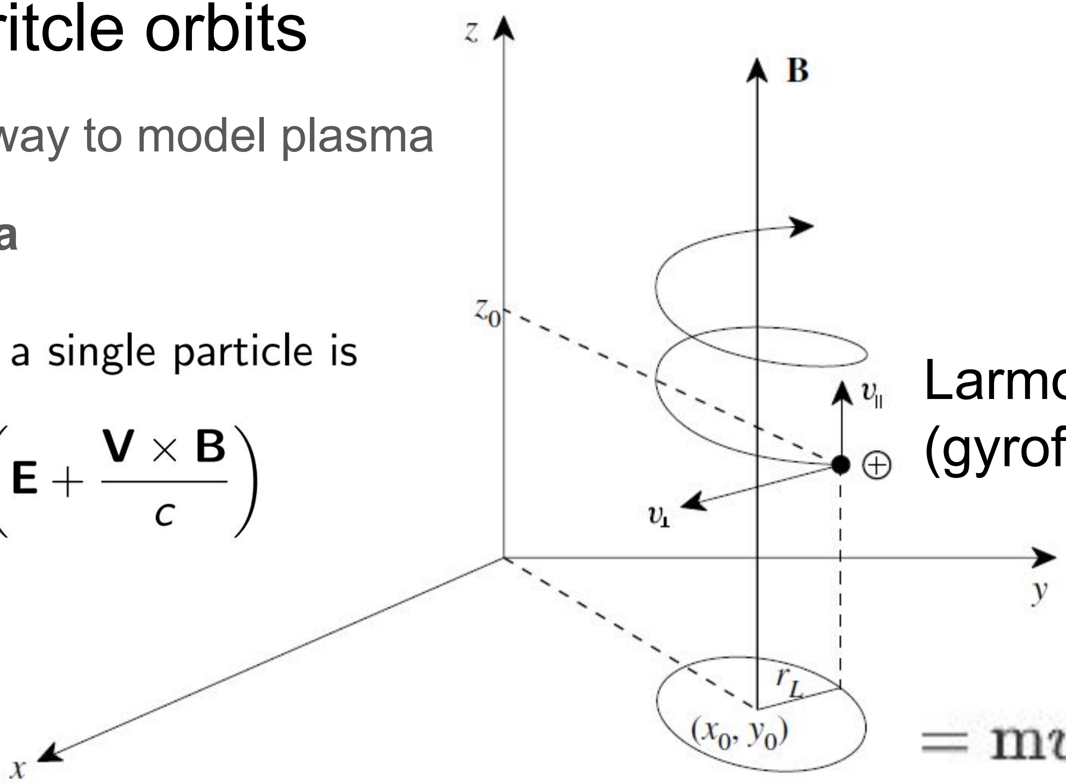
the most fundamental way to model plasma

for **low density plasma**

The Lorentz force acting on a single particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}\right)$$

in cgs unit



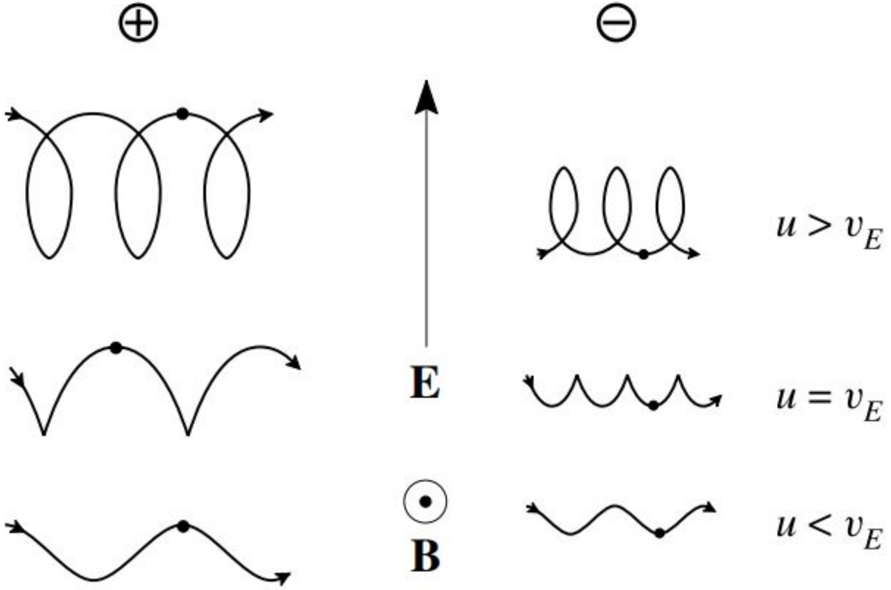
Larmor radius
(gyrofrequency)

$$= mv_\perp / qB.$$

paritcle orbits: (ExB) drift

drift velocity:

$$\mathbf{v}_\mathrm{E} = (\mathbf{E} \times \mathbf{B})/B^2$$



increase of E

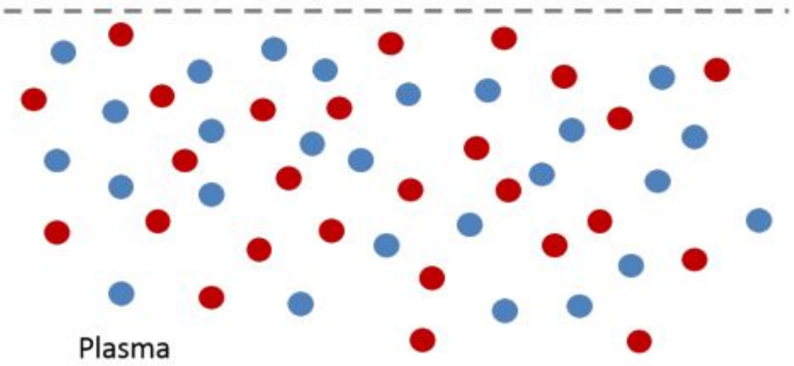
plasma

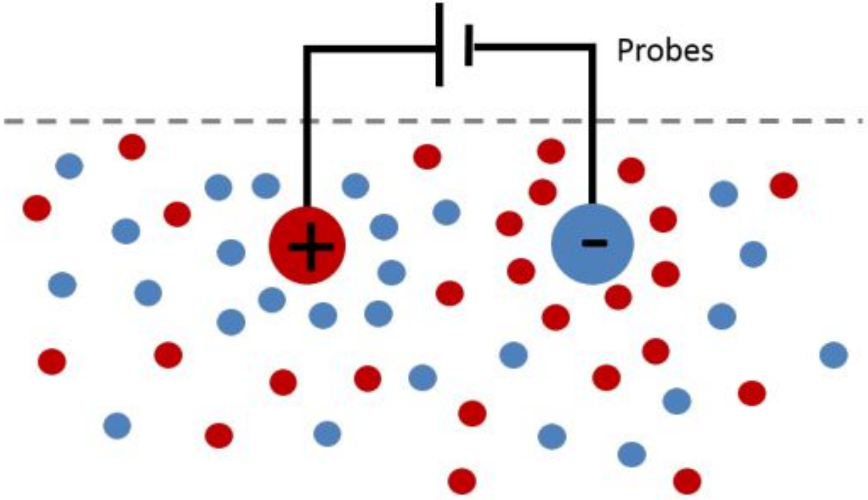
ionized gas: a state of matter comprising charged particles

A plasma is a quasi-neutral gas consisting of positive and negative charged particles (usually ions & electrons)

A stricter definition of a plasma is a gas where there are enough freed electrons and ions that they act collectively.

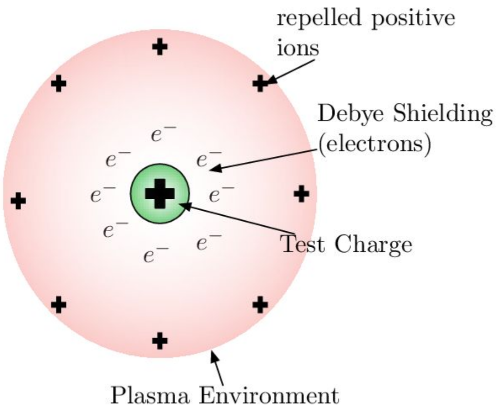
charge imblances may exist only over a distance (**Debye length**) and a period of time (inverse of **plasma frequency**)



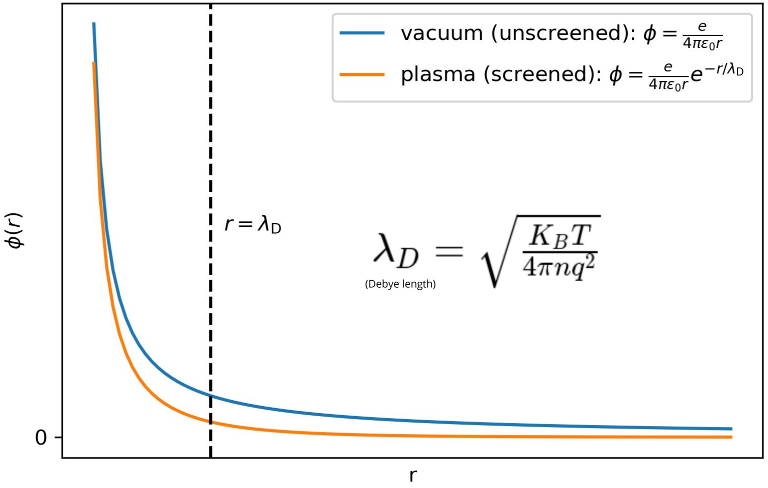


Two important parameters in plasma physics

electron Debye length λ_D : a measure of the distance over which the electric potential of a point charge is significantly influenced by the surrounding charges.



number of electrons in a “Debye cube" or “Debye sphere”: $N_D(\sim n\lambda_D^3)$



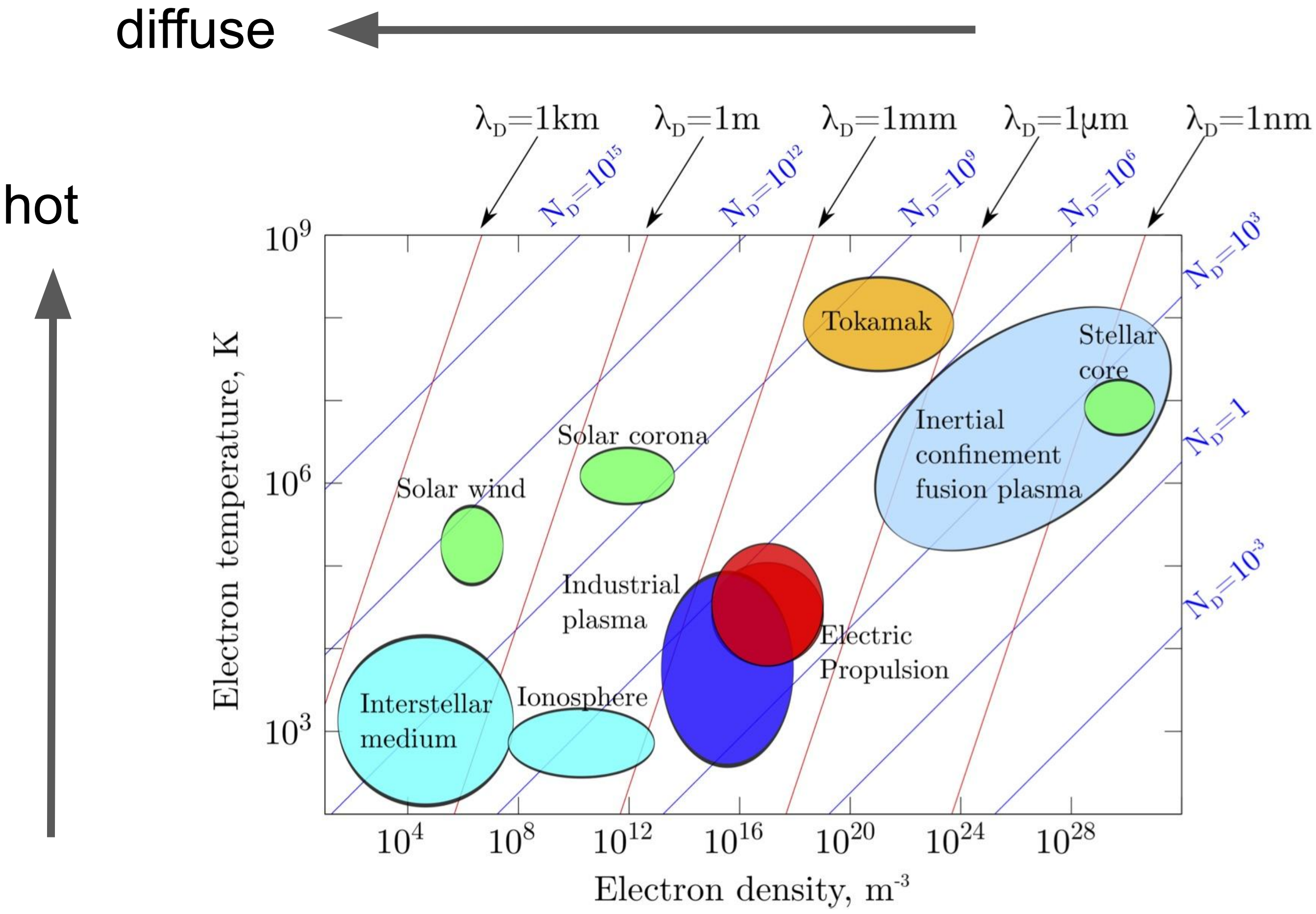
The condition for an ionized gas to be considered a plasma is $N_D >> 1$

many charged particles within a Debye cube.

Plasma exist **wide range of number densities and temperatures**

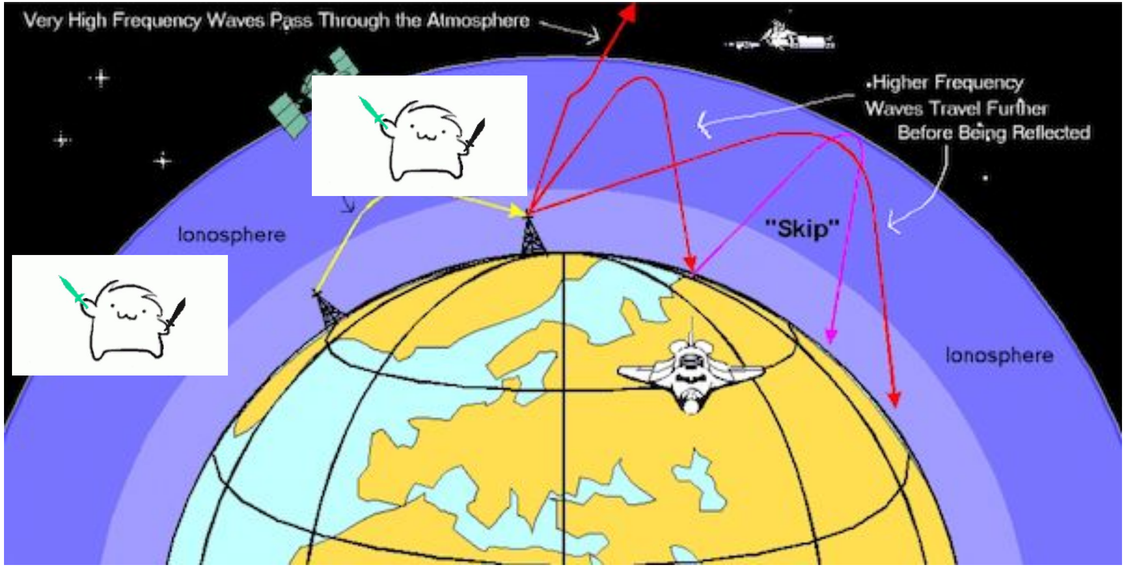
Table 1.1. *Approximate values of parameters across the plasma universe.*

Plasma	n (m^{-3})	T (keV)	B (T)	ω_{pe} (s^{-1})	λ_D (m)	$n\lambda_D^3$	ν_{ei} (Hz)
Interstellar	10^6	10^{-5}	10^{-9}	$6 \cdot 10^4$	0.7	$3 \cdot 10^5$	$4 \cdot 10^8$
Solar wind (1 AU)	10^7	10^{-2}	10^{-8}	$2 \cdot 10^5$	7	$4 \cdot 10^9$	10^{-4}
Ionosphere	10^{12}	10^{-4}	10^{-5}	$6 \cdot 10^7$	$2 \cdot 10^{-3}$	10^4	10^4
Solar corona	10^{12}	0.1	10^{-3}	$6 \cdot 10^7$	0.07	$4 \cdot 10^8$	0.5
Arc discharge	10^{20}	10^{-3}	0.1	$6 \cdot 10^{11}$	$7 \cdot 10^{-7}$	40	10^{10}
Tokamak	10^{20}	10	10	$6 \cdot 10^{11}$	$7 \cdot 10^{-5}$	$3 \cdot 10^7$	$4 \cdot 10^4$
ICF	10^{28}	10	—	$6 \cdot 10^{15}$	$7 \cdot 10^{-9}$	$4 \cdot 10^3$	$4 \cdot 10^{11}$


diffuse
hot
λD=1km
λD=1m
λD=1mm
λD=1μm
λD=1nm
ND=10^15
ND=10^12
ND=10^9
ND=10^6
ND=10^3
ND=1
ND=10^-3
Tokamak
Stellar core
Inertial confinement fusion plasma
Solar corona
Solar wind
Industrial plasma
Electric Propulsion
Interstellar medium
Ionosphere
10^9
10^6
10^3
10^4
10^8
10^12
10^16
10^20
10^24
10^28
Electron temperature, K
Electron density, m^-3

plasma frequency (~9000 x $n^{1/2}$Hz for electron)

- the frequency at which the electrons in the plasma naturally oscillate relative to the ions
- For the ionosphere, plasma frquency ~10^7Hz
- f<10^7Hz: reflected by ionosphere



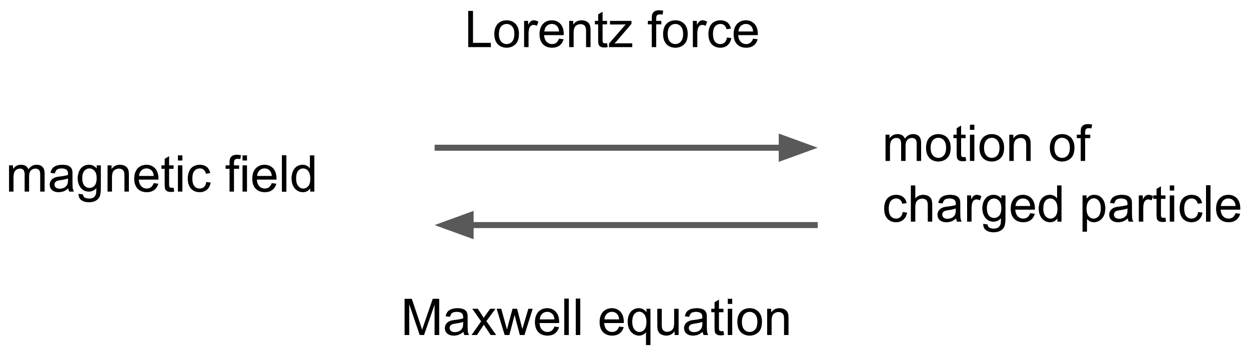
Credit: NASA/GSFC

MHD describes:

the “slow” evolution of an electrically conducting fluid, and a region >> Debye length, Larmor radius

MHD flow:
a quasi-neutral gas of charged (ionized) and neutral particles which exhibits collective behaviors

collective behavior of fluids composted of charged particles (but electrically neutral!)



*magnetic forces on the particles in the fluid are not isotropic

some initial guess

adding Lorentz force to momentum equation

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} = \frac{\mathbf{J}\times\mathbf{B}}{c} - \nabla p$$

adding Ohm's law (J) to close the set of equations

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

conductivity

MHD: single fluid approach

$$\rho = m^+ n^+ + m^- n^-$$

$$v = \frac{m^+ n^+ v^+ + m^- n^- v^-}{m^+ n^+ + m^- n^-}$$

some initial guess

- In lab: apply presence of E and/or B
- astrophysical: generated by the motion and distribution of the charged particles

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

relation to related E B to the charge and current ⇒ Maxwell's equation

some initial guess

from (3): E/B~L/T~u

from (4):

RHS 2nd term/LHS ~ u^2/c^2

nonrelativistic flow (u<<c): ignored terms related to E!

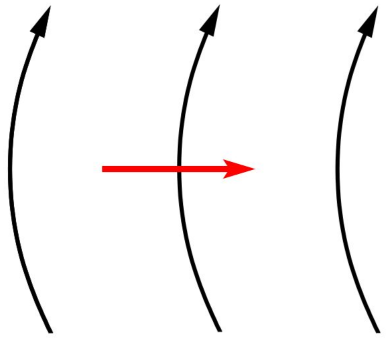
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

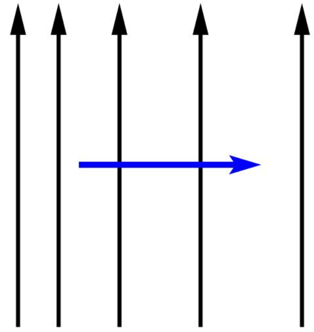
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}}$$

$$
\begin{aligned}
\underbrace{\frac{\mathbf{J} \times \mathbf{B}}{c}}_{\text{Lorentz force}} &= \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \\
&= \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{\sim\text{magnetic tension}} - \underbrace{\nabla \left(\frac{B^2}{8\pi} \right)}_{\sim\text{magnetic pressure}} \qquad (20)
\end{aligned}
$$





Lab frame:

finite

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

plasma frame:
E'=0

ideal MHD:
infinite/perfect conductivity

$$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0 \text{ and } \frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B})$$

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

finite conductivity

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

magnetic diffusivity

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta\nabla^2 B$$

(induction equation)

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B$$

$$R_m = \frac{UL}{\eta} \sim \frac{\text{induction}}{\text{diffusion}}$$

L - Typical length scale of the flow   U - Typical Velocity scale of the flow

R_m- Reynolds Magnetic Number   η - Magnetic Diffusivity

usually >>1 in astrophysics
(ideal MHD is a good approximation)

magnetic tension, magnetic pressure

Lorentz force

magnetic field

motion of
charged particle

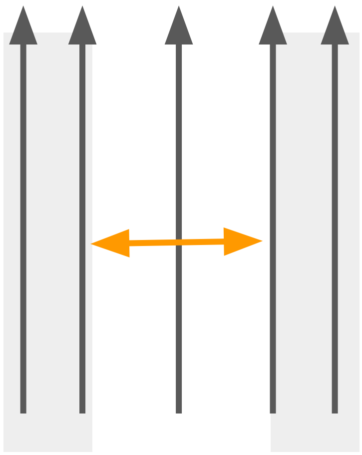
Maxwell equation + Ohm's law

$$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0$$ (ideal MHD)

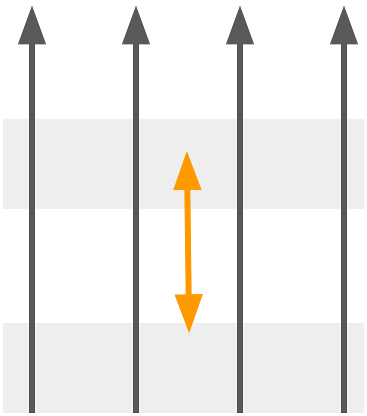
three MHD waves

fast magnetosonic wave

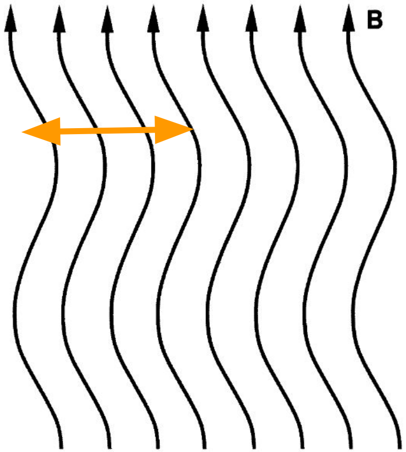
B



slow magnetosonic wave



Aflven wave



density enhancement

*from “Astrophysical flows” by Pringle and King

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$
Ampere's law	$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0$
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^{\gamma}} \right) = 0$